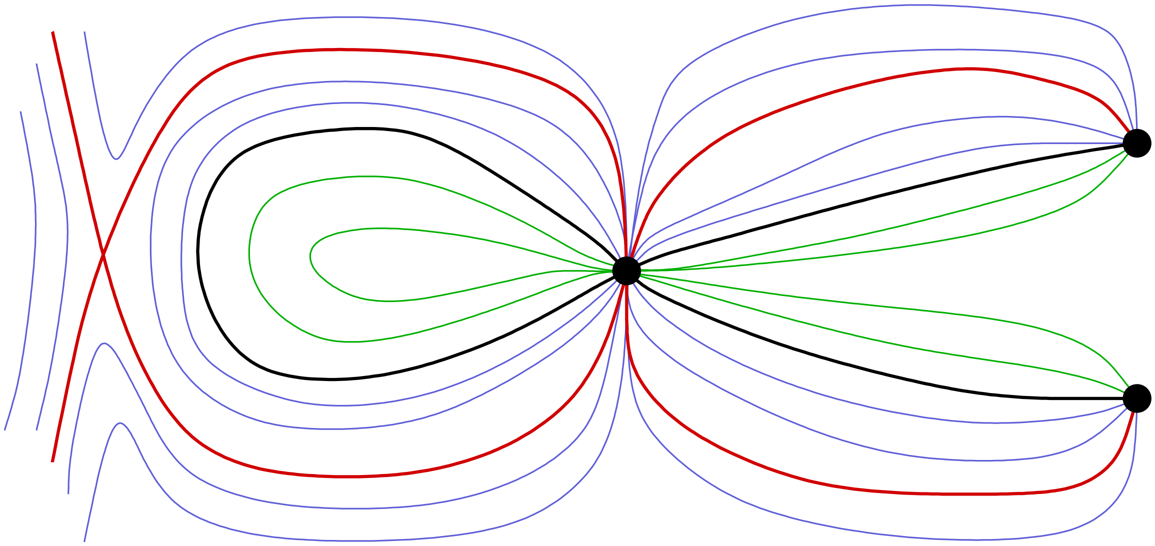
Some Miraculous Things about Holomorphic Curves in Low Dimensions



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Motivation: Transversality Problems

Enumerative invariants in an ideal world:

M = manifold, X = auxiliary data on M,

$$\Rightarrow \text{equation (PDE): } \boxed{F_X(u) = 0}$$

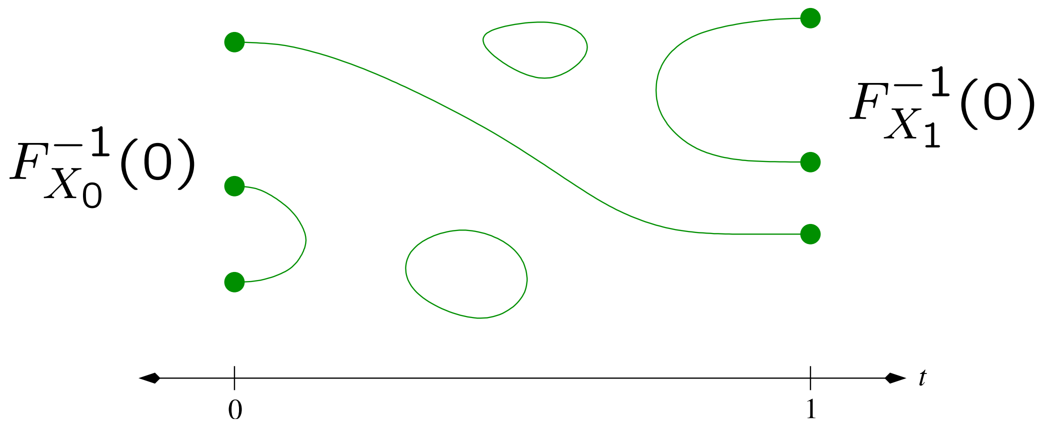
Define "invariant" $I(M, X) := \# F_X^{-1}(0)$,
for generic X, then prove...

"Theorem": $I(M, X)$ doesn't depend on X.

"Proof": For generic homotopies $\{X_t\}_{t \in [0,1]}$,

$$\mathcal{M}_{[0,1]} := \{(t, u) \mid t \in [0, 1],\ F_{X_t}(u) = 0\}$$

is a compact smooth manifold with boundary.



For example: J–holomorphic curves

(W, ω) = symplectic manifold
J = compatible almost complex structure
(Σ, j) = Riemann surface

$\mathcal{M} := \{u : \Sigma \to W \mid Tu \circ j = J \circ Tu\}$/reparam.

Analysis: $\boxed{\mathcal{M} \cong \bar{\partial}_J^{-1}(0)/\text{symmetries}}$, where $\bar{\partial}_J$ is a smooth Fredholm section of a Banach space bundle.

$D_u :=$ the linearization of $\bar{\partial}_J$ at u.

We say $u : (\Sigma, j) \to (W, J)$ in $\mathcal{M}$ is **regular** if D_u is surjective.

$\Rightarrow$ near u, $\bar{\partial}_J^{-1}(0)$ is a smooth manifold of dimension = Fredholm index of D_u.

$$\boxed{\text{ind}(u) := \text{“}\dim \mathcal{M} \text{ near } u\text{”}}$$

An almost wonderful fact:

Theorem: For generic J, every simple curve $u \in \mathcal{M}$ is regular.

"Simple" $=$ *"not multiply covered"*:

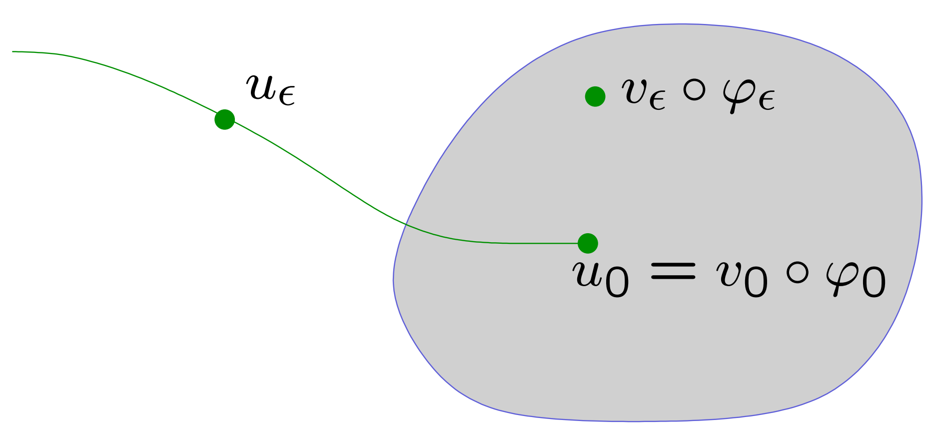
$$u \neq v \circ \varphi,$$

where $\varphi : (\Sigma, j) \to (\Sigma', j')$ is a branched cover with $\deg(\varphi) \geq 2$.

$\mathcal{M}$ is not generally smooth:
regularity fails at multiple covers.

How bad is this?

E.g. sometimes "$\dim \partial\mathcal{M} > \dim \mathcal{M}$":



Possible transversality solutions:

1. Abstract perturbations: $\bar{\partial}_J(u) = \varepsilon$.

   This is the only way to do things in full generality, but it has some disadvantages:

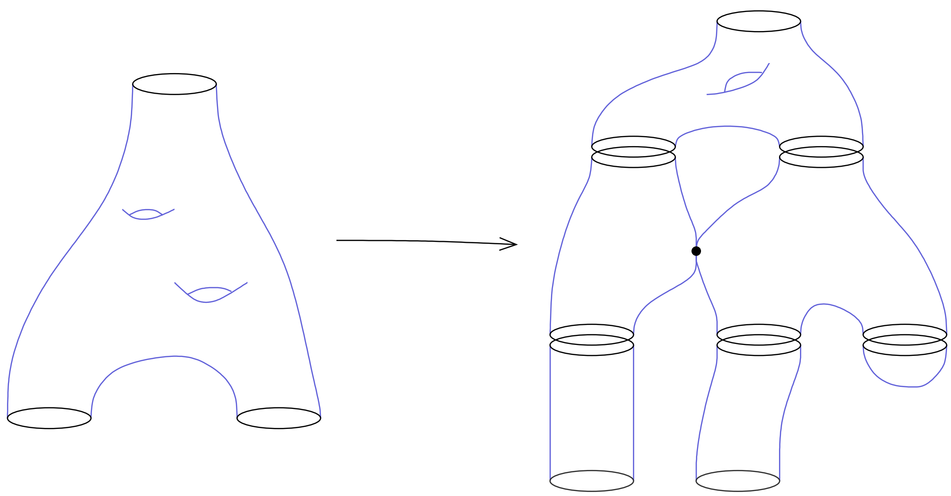
   - Analysis requires new methods, e.g. *polyfold theory*
   - Destroys nice geometric properties, such as *positivity of intersections*

2. Hope for a **miracle**!
   For curves that are "nice" geometrically, exploit these properties to show:

   "Nice curves live in nice moduli spaces."

Recall: the compactification of $\mathcal{M}$

$\overline{\mathcal{M}} :=$ {nodal J–holomorphic buildings}



Goal:

Show that if $u \in \mathcal{M}$ is "nice", so is its connected component $\overline{\mathcal{M}}_u \subset \overline{\mathcal{M}}$

Foliations and Miracles of Analysis

I. Symplectizations

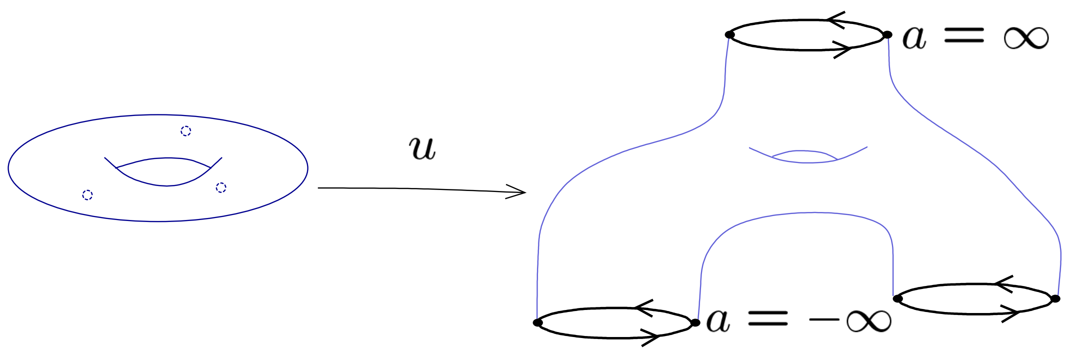
(M, λ) = contact 3–manifold
X_λ = Reeb vector field on M

On $W := \mathbb{R} \times M$, choose an $\mathbb{R}$–invariant almost complex structure $\tilde{J}$

Consider punctured $\tilde{J}$–holomorphic curves

$$\tilde{u} = (a, u) : \dot{\Sigma} \to \mathbb{R} \times M$$

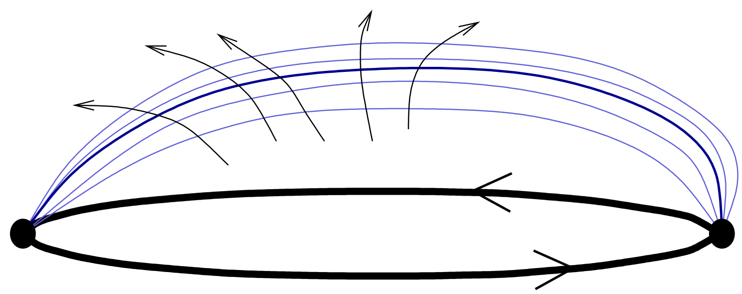
asymptotic to closed Reeb orbits.



We say $\tilde{u} = (a, u)$ is **nicely embedded** if $u : \dot{\Sigma} \to M$ is an embedding *into the* 3–*manifold*.

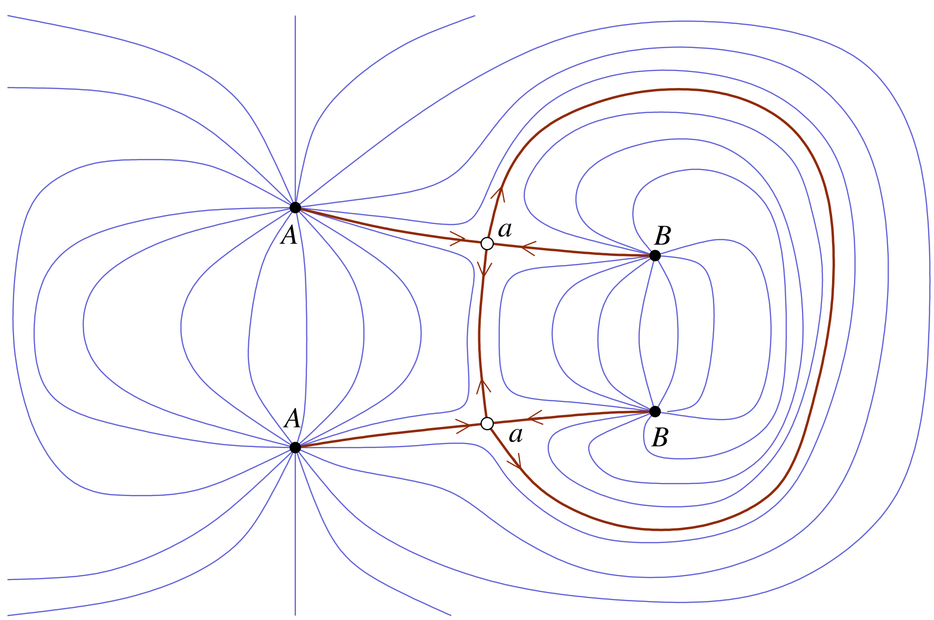
Nicely embedded $\Rightarrow$

- If $\mathrm{ind}(\tilde{u}) = 2$, nearby curves foliate a neighborhood of $u(\dot{\Sigma}) \subset M$.



- If $\mathrm{ind}(\tilde{u}) = 1$, $u(\dot{\Sigma}) \subset M$ appears isolated.

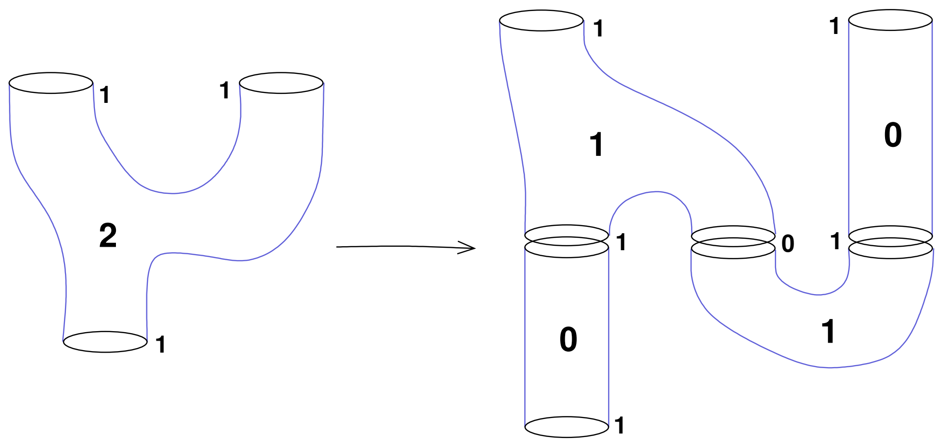
These can form *"finite energy foliations"*:



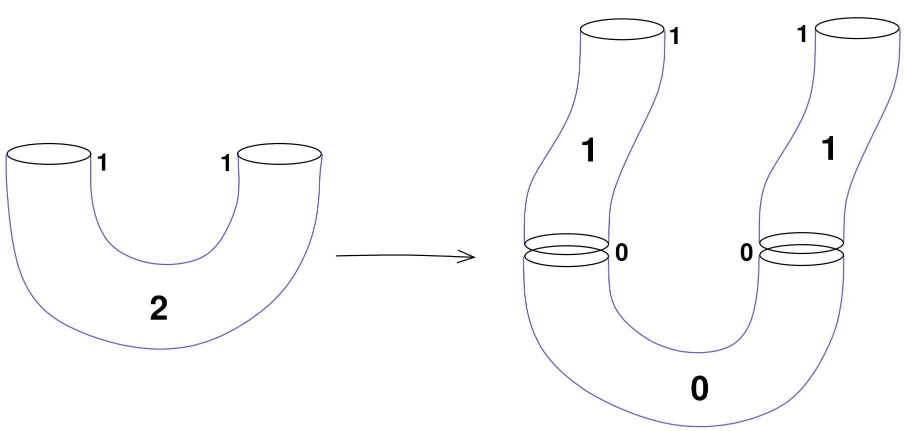
Theorem (arXiv:math/0703509)
If $\tilde{u}$ is nicely embedded, then all buildings in $\overline{\mathcal{M}}_{\tilde{u}}$ consist of nicely embedded curves and trivial cylinders over orbits.

Corollary: for generic $\tilde{J}$, all curves appearing in $\overline{\mathcal{M}}_{\tilde{u}}$ are regular

$\Rightarrow$ $\overline{\mathcal{M}}_{\tilde{u}}$ is a compact manifold with boundary.

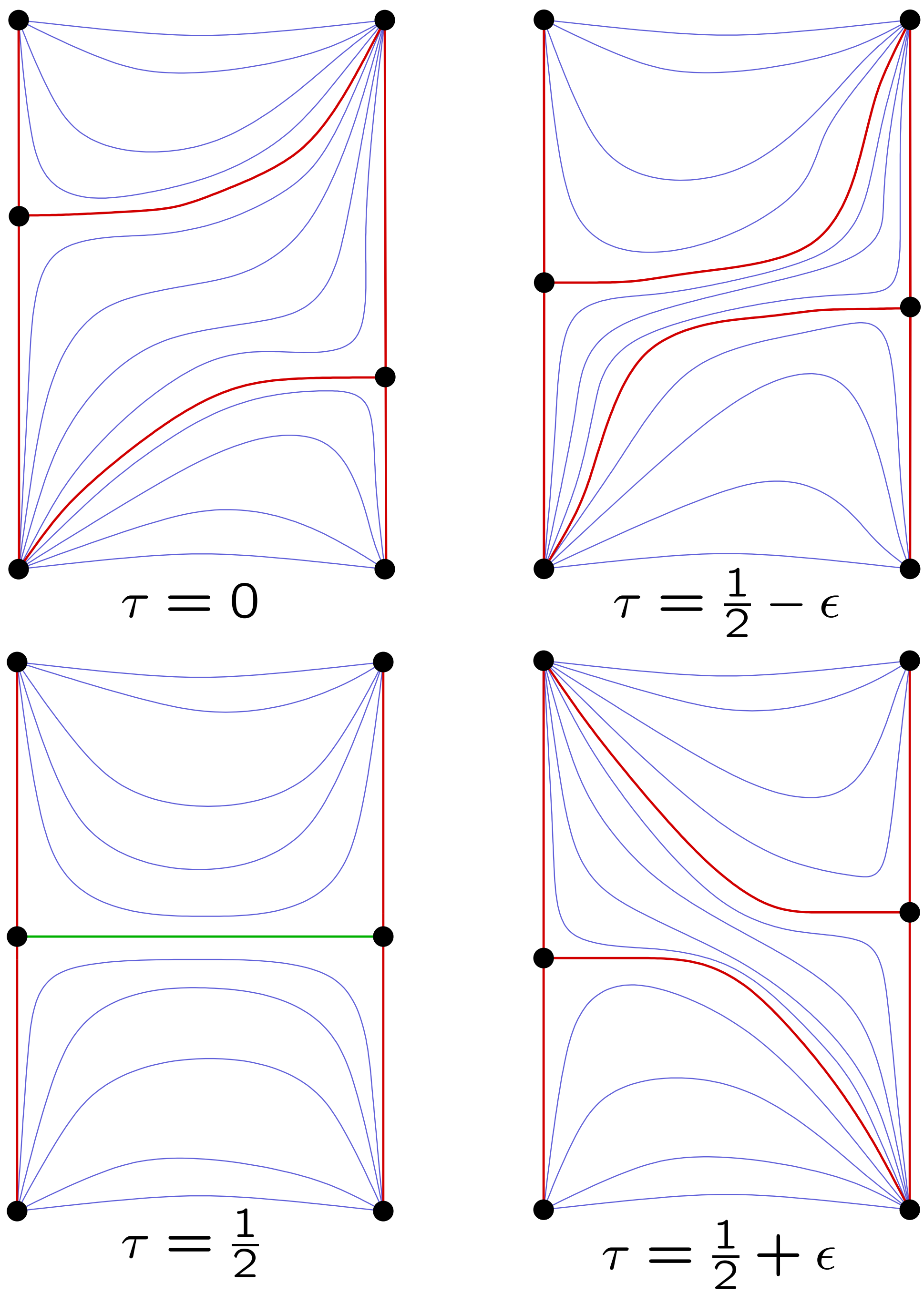


An example with non-generic J:



Application:

homotopies of finite energy foliations

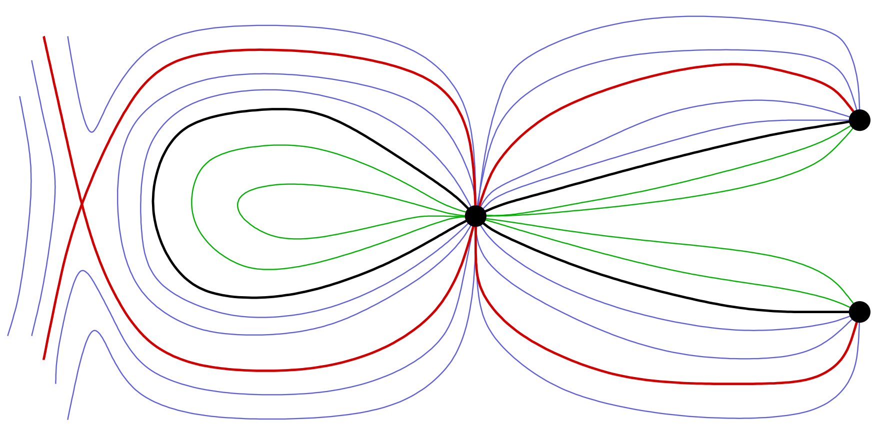


II. The closed case

$(W, J) =$ closed almost complex 4–manifold,
$(\Sigma, j) =$ closed Riemann surface

$u : (\Sigma, j) \to (W, J)$ nicely embedded $\Longleftrightarrow$
embedded, $\mathrm{ind}(u) = 2$ and $u \bullet u = 0$

(Can also generalize for immersed curves with fixed double points.)



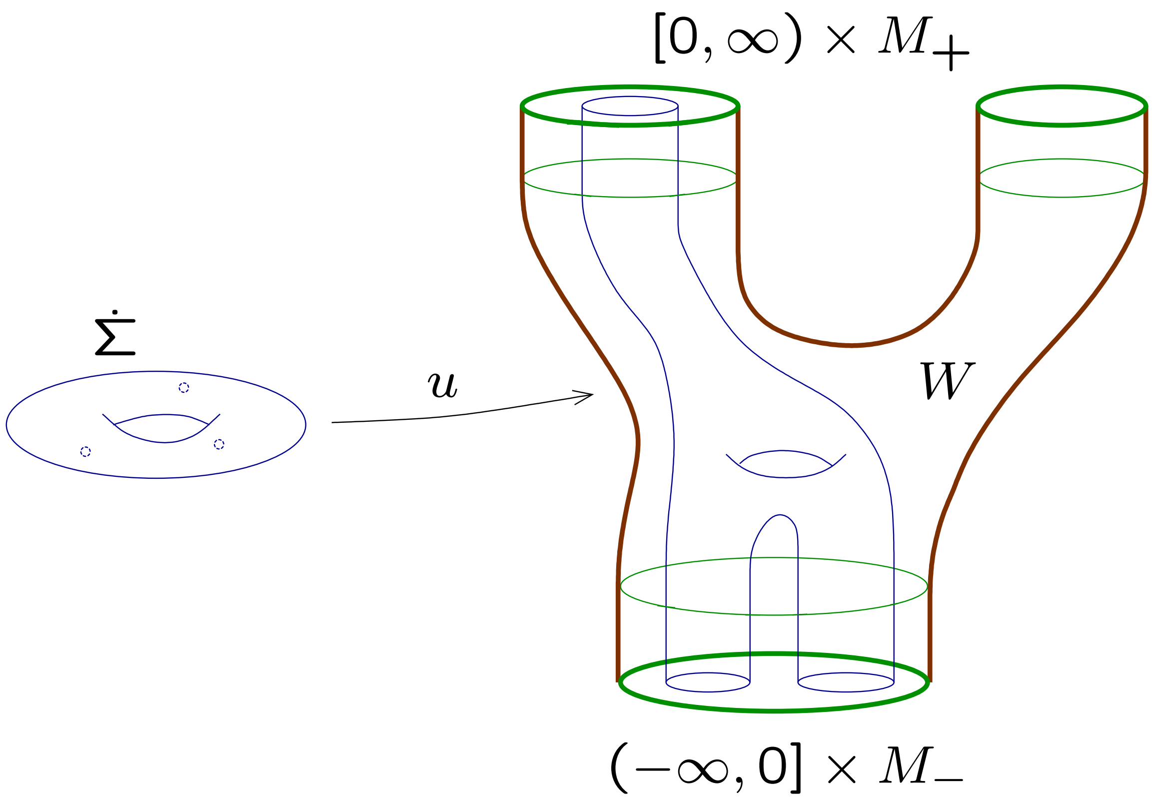
Theorem ($\Leftarrow$ *adjunction formula*):
u nicely embedded and J generic $\Rightarrow$
non-embedded curves in $\overline{\mathcal{M}}_u$ are nodal, with two embedded, transverse index 0 pieces.

Corollary: regularity for generic J

$\Rightarrow$ *(by gluing)* $\overline{\mathcal{M}}_u$ is a closed manifold.

III. The general (cobordism) case

$(W, J) =$ 4–manifold with cylindrical ends
$(\dot{\Sigma}, j) =$ punctured Riemann surface



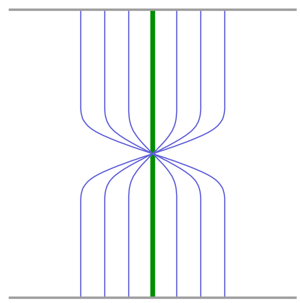
Conjecture: u nicely embedded $\Rightarrow$
$\overline{\mathcal{M}}_u$ is a smooth object (in some sense)

Partial result (arXiv:0802.3842):
u nicely embedded and J generic $\Rightarrow$
$\mathcal{M}_u$ is a smooth **orbifold**, with isolated singularities that consist of **unbranched** multiple covers over embedded index 0 curves.

Why orbifolds?

A lower-dimensional example:

$\mathcal{M}$:= smooth 1–parameter family of (unparametrized) closed orbits



Regularity $\Rightarrow$
{parametrized orbits} $\cong$ smooth surface
(Möbius strip)

$\Rightarrow \mathcal{M} \cong \text{surface}/S^1$.

Middle orbit has stabilizer $\mathbb{Z}_2$ under S^1–action,
$\Rightarrow \mathcal{M} \cong$ open subset of $\mathbb{R}/\mathbb{Z}_2$.

symmetry $\Leftrightarrow$ orbifold singularities

For holomorphic curves:

$$\mathcal{M} \cong \bar{\partial}_J^{-1}(0)/\text{symmetries}$$

u regular $\Rightarrow$ $\bar{\partial}_J^{-1}(0)$ is a manifold near u.

Stabilizer of u is

$$\text{Aut}(u) := \{\varphi : (\Sigma, j) \xrightarrow{\sim} (\Sigma, j) \mid u = u \circ \varphi\}.$$

This can be nontrivial if u is multiply covered.

$\therefore$ Regularity $\Rightarrow$

$$\boxed{\begin{array}{c} \text{nbhd}(u) \subset \mathcal{M} \\ \cong \\ \text{open subset} \subset \mathbb{R}^{\text{ind}(u)} / \, \text{Aut}(u). \end{array}}$$

Task: prove regularity for all curves in $\mathcal{M}_u$, *including the multiple covers*.

Idea of Proof

Define the *normal Chern number*:

$$\boxed{c_N(u) := c_1(u^*TW) - \chi(\Sigma)}$$

Then the adjunction formula is

$$u \bullet u = 2\delta(u) + c_N(u),$$

$\Rightarrow$ nicely embedded curves have $c_N(u) = 0$.

The following *automatic transversality* result for closed curves holds in dimension four for *all* (not just generic) J:

Theorem (Hofer-Lizan-Sikorav):
If $u : \Sigma \to W^4$ is immersed and satisfies

$$\boxed{\mathrm{ind}(u) > c_N(u)}$$

then u is regular.

$\therefore$ When $u_j \to u = v \circ \varphi$, regularity follows if u is immersed. Indeed, one can show:
(1) v is embedded,
(2) φ is unbranched.

$\therefore$ Multiple covers can appear, but **only the harmless type!**

Generalizing Hofer-Lizan-Sikorav:

Remark 1: One can define $c_N(u)$ for punctured curves so that it suitably generalizes "c_1 of the normal bundle".

Remark 2: There exists a "tangent-normal" splitting

$$u^*TW = T_u \oplus N_u$$

even if u has critical points. Here

$$c_1(N_u) = c_N(u) - \#\,\mathrm{Crit}(u).$$

One can then prove:

Theorem ("generalized automatic $\pitchfork$"):
If $u : \dot{\Sigma} \to W^4$ satisfies

$$\boxed{\mathrm{ind}(u) > c_N(u) + \#\,\mathrm{Crit}(u),}$$

then u is regular.

Remark 3: That's nice, but we won't use it.

Remark 4: Let D_u^N := the *normal part* of D_u. Then it turns out,

$$\boxed{\text{dim ker } D_u = \text{dim ker } D_u^N + 2\,[\#\,\text{Crit}(u)]}$$

Now if u_j are nicely embedded, $u_j \to u = v \circ \varphi$, v is embedded and φ is branched, this implies

$$\text{dim ker } D_u = 2\,[\#\,\text{Crit}(\varphi)]\,.$$

This gives a contradiction, because all u' near u are then of the form

$$u' = v \circ \varphi'$$

for φ' near φ.