

ON SYMPLECTIC FILLINGS OF SPINAL OPEN BOOK DECOMPOSITIONS I: GEOMETRIC CONSTRUCTIONS

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ABSTRACT. A spinal open book decomposition on a contact manifold is a generalization of a supporting open book which exists naturally e.g. on the boundary of a symplectic filling with a Lefschetz fibration over any compact oriented surface with boundary. In this first paper of a two-part series, we introduce the basic notions relating spinal open books to contact structures and symplectic or Stein structures on Lefschetz fibrations, leading to the definition of a new symplectic cobordism construction called *spine removal* surgery, which generalizes previous constructions due to Eliashberg [Eli04], Gay-Stipsicz [GS12] and the third author [Wen13b]. As an application, spine removal yields a large class of new examples of contact manifolds that are not strongly (and sometimes not weakly) symplectically fillable. This paper also lays the geometric groundwork for a theorem to be proved in part II, where holomorphic curves are used to classify the symplectic and Stein fillings of contact 3-manifolds admitting a spinal open book with a planar page.

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0. INTRODUCTION

The present paper is the first in a two-part series aimed at generalizing the well-known interplay between contact structures with supporting open book decompositions and their fillings by symplectic or Stein manifolds with symplectic Lefschetz fibrations. We can point to at least two specific previous applications of open books in contact topology as inspiration for this project:

- (1) In [Wen10, NW11], the third author proved that for every contact 3-manifold supported by a planar open book, the deformation classes of its symplectic fillings are in bijective correspondence to the diffeomorphism classes of Lefschetz fibrations over \mathbb{D}^2 that fill the open book. Some version of this statement is true moreover for all of the usual notions of symplectic fillability (i.e. weak, strong, Liouville and Stein), thus proving that for planar contact manifolds, they are all equivalent. The problem of classifying fillings for such contact manifolds was reduced in this way to a factorization problem on the mapping class group of surfaces, cf. [PV10, Pla12, Wan12, KL16, Kal].
- (2) In [Eli04], Eliashberg used non-exact symplectic 2-handles attached along the binding of an open book to construct symplectic caps for all closed contact 3-manifolds. This served among other things as an ingredient in Kronheimer-Mrowka's proof of Property P [KM04], and it was later generalized to various forms of non-exact symplectic cobordism between contact manifolds, cf. [Gay06, GS12, Wen13b].

The motivating question behind the present project was as follows: what structure naturally arises on the convex boundary of a Lefschetz fibration with exact symplectic fibers over a surface with boundary other than \mathbb{D}^2 ? Spinal open books are the answer to this question, and we will show that they give rise to far-reaching generalizations of both of the results mentioned above. One example of the first type appeared already in [Wen10], where the symplectic fillings of \mathbb{T}^3 were classified in terms of Lefschetz fibrations over the annulus $[-1, 1] \times S^1$. This was proved using methods from the low-dimensional theory of J -holomorphic curves, and the aim of the sequel to this paper [LVW] will be to push those techniques as far as they can reasonably be pushed.

Here is an initial sketch of the main idea. Roughly speaking, a spinal open book decomposes a 3-manifold M into two (possibly disconnected) pieces, called the *paper* M_P and the *spine* M_Σ , where M_P consists of families of *pages* fibering over S^1 , M_Σ is an S^1 -fibration over some collection of compact oriented surfaces, and the boundaries of fibers in M_P consist of fibers in M_Σ (see Figure 1). The usual notion of open books is recovered if one takes the base of the fibration on M_Σ to be a disjoint union of disks (see Example 1.9); similarly, allowing annuli in the base produces the notion of *blown up summed open books* (Example 1.11), which were studied in [Wen13a]. One of the main results of [LVW] can be summarized as follows (see §1 below for the pertinent definitions):

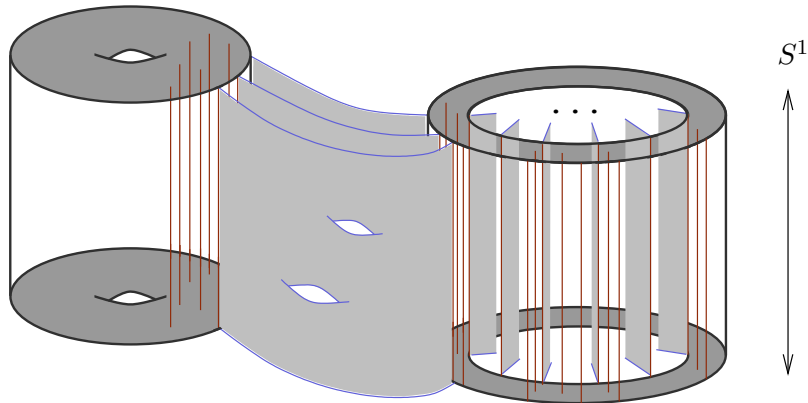


FIGURE 1. A spinal open book with two spine components, which are S^1 -fibrations over a genus 1 surface with one boundary component and an annulus respectively. They are connected to each other by an S^1 -family of pages with genus 2, and we can also see a fragment of a second S^1 -family of pages attached to the annular spine component.

Theorem A ([LVW]). *Suppose (M, ξ) is a closed contact 3-manifold containing a domain M_0 on which ξ is supported by an amenable spinal open book π that has a planar page in its interior. If (M, ξ) admits a weak filling that is exact on the spine of π , then $M = M_0$, and the set of weak symplectic fillings of (M, ξ) that are exact on the spine is, up to symplectic deformation equivalence, in one-to-one correspondence with the set of Lefschetz fibrations (up to diffeomorphism) that match π at their boundaries. Moreover, every such filling can be deformed to a blowup of a Stein filling.*

To focus for a moment on Stein fillings in particular: most previous results classifying Stein fillings have classified them up to diffeomorphism or symplectic deformation, the only exceptions we are aware of being results of Eliashberg [Eli90, CE12] and Hind [Hin00, Hin03], which achieved uniqueness up to Stein deformation equivalence for fillings of S^3 , connected sums of $S^1 \times S^2$, and certain lens spaces. In these examples, the classification up to Stein deformation matches the classification up to symplectic deformation, and we will see that this is not a coincidence—it can be seen as a symptom of a general *quasiflexibility* phenomenon for Stein surfaces:

Theorem B ([LVW]). *Suppose W is a compact 4-manifold with boundary, admitting two Stein structures J_0 and J_1 such that (W, J_0) is compatible with a Lefschetz fibration (over an arbitrary compact oriented surface) with fibers of genus zero. Then J_0 and J_1 are Stein homotopic if and only if their induced symplectic structures are homotopic as symplectic structures convex at the boundary.*

Note that the symplectic deformation in this statement need not be in a fixed cohomology class—in particular, quasiflexibility is a very different phenomenon from the familiar relationship between Stein and Weinstein structures (cf. [CE12]).

While the results quoted above require holomorphic curve techniques, this first paper in the series will focus on the less analytical but more geometric aspects of the theory of spinal open books. We will start by giving natural constructions of contact structures supported by spinal open books and symplectic or Stein structures related to them. The most subtle

of these results pertains specifically to Stein (or equivalently Weinstein) structures, and gives a verifiable criterion in terms of Lefschetz fibrations for two Stein structures to be Stein homotopic. This will serve in [LVW] as an essential ingredient for the classification of Stein fillings up to Stein deformation and the proof of Theorem B. The result is most easily stated in terms *almost Stein structures*, which are pairs (J, f) consisting of an almost complex structure J and a J -convex function f . Here J is not required to be integrable, and f need not be constant at the boundary, thus they do not immediately define a Stein structure, but if we assume the Liouville vector field dual to $-df \circ J$ is outwardly transverse at the boundary, then (J, f) nonetheless determines a Weinstein structure canonically up to Weinstein homotopy (see §1.1.3).

The following theorem can be interpreted as saying that the Stein homotopy class of a Stein structure can be deduced from a Lefschetz fibration if it satisfies fairly strict compatibility conditions near the boundary but a minimum of reasonable conditions in the interior—this result is well suited in particular to the scenario in which fibers of a Lefschetz fibration are J -holomorphic curves.

Theorem C (see Theorem 3.1). *Suppose $\Pi : E \rightarrow \Sigma$ is a Lefschetz fibration whose regular fibers and base are each compact oriented surfaces with nonempty boundary, and write*

$$\partial_v E := \Pi^{-1}(\partial\Sigma), \quad \partial_h E := \bigcup_{z \in \Sigma} \partial E_z.$$

For $\tau = 0, 1$, assume J_τ is an almost complex structure on E and $f_\tau : E \rightarrow \mathbb{R}$ is a smooth J_τ -convex function such that the following conditions are satisfied:

- (1) J_τ preserves the vertical subbundle of TE and is compatible with its orientation;
- (2) f_τ is constant on the boundary components of every fiber;
- (3) The Liouville form $\lambda_\tau := -df_\tau \circ J_\tau$ restricts to both $\partial_v E$ and $\partial_h E$ as contact forms, the induced Reeb vector field on $\partial_h E$ is tangent to the fibers, and its flow preserves the maximal J_τ -complex subbundle of $T(\partial_h E)$;
- (4) There exists a complex structure j_τ on Σ and an open neighborhood $\mathcal{U} \subset \Sigma$ of $\partial\Sigma$ such that the Cauchy-Riemann equation $T\Pi \circ J_\tau = j_\tau \circ T\Pi$ is satisfied on $E|_{\mathcal{U}}$ and $\partial_h E$.

Then the Weinstein structures on E (after smoothing the corners) determined by (J_0, f_0) and (J_1, f_1) are Weinstein homotopic.

With this groundwork in place, we will then introduce a new construction of non-exact symplectic cobordisms that generalizes previous results from [Eli04, GS12, Wen13b] and arises from a natural topological operation on spinal open books called *spine removal surgery*. An informal version of the result can be stated as follows:

Theorem D (see Theorem 1.25). *Assume (M, ξ) is a contact 3-manifold supported by a spinal open book π , $\Sigma^{\text{rem}} \times S^1 \cong M^{\text{rem}} \subset M_\Sigma$ is an open and closed subset of the spine of π , and $\tilde{\pi}$ is a spinal open book on a contact 3-manifold $(\tilde{M}, \tilde{\xi})$ defined by deleting M^{rem} from M and capping off all adjacent boundary components of pages of π by disks. Then there exists a symplectic cobordism with strongly concave boundary (M, ξ) and weakly convex boundary $(\tilde{M}, \tilde{\xi})$, defined by attaching the “handle” $\Sigma^{\text{rem}} \times \mathbb{D}^2$ with a product symplectic structure along $\Sigma^{\text{rem}} \times S^1 \cong M^{\text{rem}}$.*

Special cases of this operation were used in [Wen13b] to construct non-exact symplectic cobordisms between pairs of contact 3-manifolds that do not admit exact ones, e.g. it showed that all of the known examples of contact 3-manifolds with finite orders of algebraic torsion

(cf. [LW11]) are symplectically cobordant to overtwisted ones. We will use the general version in this paper to prove the vast majority of cases of Theorem A for which the contact manifold turns out to be non-fillable, a result that can be interpreted as generalizing the local filling obstruction defined as *planar torsion* in [Wen13a]. A slightly different kind of application appears in [LW21], where spine removal is used to prove that contact 3-manifolds supported by planar spinal open books satisfy a universal bound on the geography of their symplectic fillings. This generalizes a previous result for the case of planar open books due to Plamenevskaya [Pla12] (see also [Kal]).

In our project we have focused specifically on dimension three, since that is where the strongest results on classification of fillings can be proved, but it should be mentioned that the theory of spinal open books has already had some impact on developments in higher-dimensional contact topology. In dimension $2n - 1$, it is natural to consider decompositions $M = M_\Sigma \cup M_P$ where M_P is a fibration of Liouville domains over a contact manifold and M_Σ is a strict contact fibration over a Liouville domain. Taking \mathbb{D}^2 and S^1 as bases produces the usual notion of open books in arbitrary dimensions, but it is sometimes also useful to allow higher-dimensional bases, e.g. the first author has observed that Bourgeois's construction [Bou02] of contact structures on $M \times \mathbb{T}^2$ can be understood as an operation replacing \mathbb{D}^2 and S^1 with $T^*\mathbb{T}^2$ and \mathbb{T}^3 as base spaces in a spinal open book (cf. [LMN19]). Working with strictly low-dimensional fibers but higher-dimensional bases, [MNW13] constructed a higher-dimensional version of a spine removal cobordism in order to establish the first examples of higher-dimensional tight contact manifolds that are not symplectically fillable. More recently, Moreno [Mor18, Mor24] and Zhou [Zhou] have used high-dimensional spinal open books to construct new examples of contact manifolds with higher-order algebraic torsion. Spinal open books also appeared in work of Acu and Moreno [AM22], which used a variant of spine removal surgery to study a higher-dimensional analogue of planar contact manifolds, and in Zhou's construction [Zhou] of contact manifolds in arbitrary dimensions admitting infinitely many distinct Weinstein fillings.

A remark on timing. While this paper is intended as the “official” introduction to spinal open books in dimension three, the project has by now been in preparation long enough for some of the fundamental notions to have appeared already in several other papers, some by the authors plus collaborators (e.g. [BV15]) and some by others (e.g. [MRW, PS]). We have tried to make sure all definitions are consistent with what has previously appeared, but in the event of any discrepancies, the present paper is meant to be definitive.

Outline of the paper. Section 1 is an extended introduction, intended to give precise versions of all the essential definitions and main results, including some definitions that are needed mainly for the classification discussion in [LVW]. Section 2 then proves the essential theorems relating spinal open books and Lefschetz fibrations to their associated deformation classes of contact and symplectic structures, and §3 proves Theorem C on Stein homotopy classes. In §4, we construct a concrete symplectic model for collar neighborhoods (in the symplectization) of a contact manifold supported by an arbitrary spinal open book, which is then used to prove the main theorem on symplectic cobordisms arising from spine removal surgery. This result is then applied in §5 to establish new criteria for nonfillability.

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1. DEFINITIONS AND RESULTS

1.1. Main definitions. In this section we give the main definitions and state precise versions of the main results of the paper.

1.1.1. Types of symplectic fillings. Throughout this paper, we assume all contact structures on oriented 3-manifolds to be *co-oriented* and *positive*, i.e. they can always be written as $\xi = \ker \alpha$ where the contact form α satisfies $\alpha \wedge d\alpha > 0$. Suppose (M, ξ) is a closed contact 3-manifold and (W, ω) is a compact connected symplectic 4-manifold with boundary. Then (W, ω) is a **weak filling** of (M, ξ) if ∂W can be identified via an orientation-preserving diffeomorphism with M such that $\omega|_{\xi} > 0$. We also say in this case that ω **dominates** ξ at the boundary, and that the boundary is **weakly convex** with respect to ω . If there additionally exists a 1-form λ near ∂W that satisfies $d\lambda = \omega$ and restricts to the boundary as a contact form for ξ under the above identification $\partial W \cong M$, then the boundary is called **convex** and (W, ω) is called a **strong filling** of (M, ξ) . We say that two weak/strong fillings (W, ω) and (W', ω') of contact manifolds (M, ξ) and (M', ξ') respectively are weakly/strongly **symplectically deformation equivalent** if there exists a diffeomorphism $\varphi : W \rightarrow W'$ and smooth 1-parameter families of symplectic structures $\{\omega_\tau\}_{\tau \in [0,1]}$ on W and contact structures $\{\xi_\tau\}_{\tau \in [0,1]}$ on M such that $\omega_0 = \omega$, $\omega_1 = \varphi^*\omega'$, $\xi_0 = \xi$, $\xi_1 = \varphi^*\xi'$, and (W, ω_τ) is a weak/strong filling of (M, ξ_τ) for each $\tau \in [0, 1]$. Note that by Gray's stability theorem, deformation equivalence implies that (M, ξ) and (M', ξ') must be contactomorphic.

Recall that a symplectic 4-manifold is said to be **minimal** if it does not contain any *exceptional spheres*, i.e. symplectically embedded 2-spheres with self-intersection number -1 . By an argument due to McDuff [McD90], minimality is invariant under (strong or weak) symplectic deformation.¹

We call (W, ω) an **exact filling** of (M, ξ) if it is a strong filling such that the 1-form λ as defined above near the boundary extends to a global primitive of ω on W . In this case $(W, d\lambda)$ is also called a **Liouville domain**, with **Liouville form** λ , which determines the **Liouville vector field** V_λ via the condition

$$\omega(V_\lambda, \cdot) = \lambda.$$

Two exact fillings are said to be **Liouville deformation equivalent** if they are strongly symplectically deformation equivalent and each of the symplectic structures in the smooth homotopy defines an exact filling. Note that for any fixed ω on a Liouville domain, the space of Liouville forms λ satisfying $d\lambda = \omega$ is convex, thus every Liouville deformation in this sense can be realized by a smooth homotopy of Liouville forms.

Finally, a **Stein filling** of (M, ξ) is a compact connected complex manifold (W, J) , also called a **Stein domain**, with oriented boundary identified with M such that $\xi \subset TM$ is the maximal complex-linear subbundle, and such that there exists a smooth function $f : W \rightarrow \mathbb{R}$ that has the boundary as a regular level set (we say that f is **exhausting**) and

¹In our context, McDuff's argument that minimality is preserved under deformations depends on the conditions we impose on ω at ∂W : these guarantee in particular that one can always make the boundary J -convex for a tame almost complex structure J , thus preventing J -holomorphic spheres from escaping the interior.

is **plurisubharmonic**. The latter means that $\lambda_J := -df \circ J$ is a Liouville form and the resulting symplectic form $\omega_J := d\lambda_J$ **tames** J , i.e.

$$\omega_J(X, JX) > 0 \text{ for all nonzero } X \in TW.$$

Two Stein fillings are **Stein deformation equivalent** if they can be identified via a diffeomorphism so that the two complex structures are homotopic through a smooth family of integrable complex structures that all admit exhausting plurisubharmonic functions.

Note that for a given J , the space of exhausting plurisubharmonic functions is convex, and the plurisubharmonicity condition is open with respect to J ; one can use these facts to show that any smooth homotopy of Stein structures can be accompanied by a smooth homotopy of exhausting plurisubharmonic functions. By the correspondence $J \mapsto \lambda_J \mapsto \omega_J$ defined above, it follows that a Stein deformation class of Stein fillings always gives rise to a canonical Liouville deformation class of exact symplectic fillings. The exact fillings arising in this way have the additional feature that their Liouville vector fields are gradient-like: indeed, any exhausting plurisubharmonic function $f : W \rightarrow \mathbb{R}$ on a Stein domain (W, J) is also a Lyapunov function for the Liouville vector field V_J dual to λ_J , thus giving (W, ω_J, V_J, f) the structure of a **Weinstein domain**. We will occasionally make use of the deep theorem from [CE12] giving a one-to-one correspondence between deformation classes of Stein domains and Weinstein domains respectively.

Remark 1.1. Strictly speaking, the function f in a Weinstein structure should always be required to be Morse (or generalized Morse in the case of deformations), but on Stein domains this can always be achieved via small perturbations of plurisubharmonic functions since the plurisubharmonicity condition is open.

1.1.2. *Spinal open books.* The following topological notion will be of central importance in this paper.

Definition 1.2. A **spinal open book decomposition** on a compact oriented 3-dimensional manifold M , possibly with boundary, is a decomposition $M = M_\Sigma \cup M_P$, where the pieces M_Σ and M_P (called the **spine** and **paper** respectively) are smooth compact 3-dimensional submanifolds with disjoint interiors such that $\partial M_\Sigma \subset \partial M_P$, carrying the following additional structure:

- (1) A smooth fiber bundle $\pi_\Sigma : M_\Sigma \rightarrow \Sigma$ with connected and oriented fibers, all of which are either disjoint from ∂M_Σ or contained in it. Here, Σ is a compact oriented surface whose connected components (called **vertebrae**²) all have nonempty boundary.
- (2) A smooth fiber bundle $\pi_P : M_P \rightarrow S^1$ with oriented fibers whose connected components (called **pages**) are each preserved by the monodromy map, have nonempty boundary and meet ∂M_P transversely. Moreover, the intersection of any fiber of π_P with M_Σ consists of fibers of π_Σ .
- (3) At each connected boundary component $T \subset \partial M$ (which is necessarily a 2-torus component of ∂M_P), there is a preferred homology class $m_T \in H_1(T)$ with the property that if $f_T \in H_1(T)$ denotes the homology class of a connected component of $\pi_P^{-1}(*) \cap T$

²The use of the bookbinding metaphor for open book decompositions was the original inspiration for our choice of the terms “spine” and “paper”, though the alternative anatomical meaning of “spine” also has some advantages. The term “vertebrae” makes sense especially when one observes that the fibration $\pi_\Sigma : M_\Sigma \rightarrow \Sigma$ is necessarily trivial, thus the spine can be foliated by vertebrae. It makes less sense perhaps in higher-dimensional analogues of spinal open books, where $\pi_\Sigma : M_\Sigma \rightarrow \Sigma$ need not always be a trivial fibration.

oriented as boundary of the fiber, then (m_T, f_T) defines a positively oriented basis of $H_1(T) \cong \mathbb{Z}^2$ for the boundary orientation of ∂M . We call m_T the **preferred meridian** at T .

We should emphasize that in the above definition, neither M nor its paper or spine is required to be connected, though pages and vertebrae are connected by definition. One can also allow the spine to be empty, in which case M must have nonempty boundary (since the pages do). We shall typically denote the full collection of data defining a spinal open book on M by

$$\pi := \left(\pi_\Sigma : M_\Sigma \rightarrow \Sigma, \pi_P : M_P \rightarrow S^1, \{m_T\}_{T \subset \partial M} \right).$$

For any connected component $\gamma \subset \partial\Sigma$, the fact that boundary components of pages are also fibers of π_Σ means that there is a well-defined map

$$(1.1) \quad \gamma \rightarrow S^1 : \phi \mapsto \pi_P(\pi_\Sigma^{-1}(\phi)).$$

This map is always a diffeomorphism for ordinary open books (see Example 1.9), but more generally it may be a finite cover.

Definition 1.3. Given the spinal open book π as described above, we define the **multiplicity** of π_P at a boundary component $T \subset \partial M_P$ as the number of distinct page boundary components that touch T . If $T \subset M_P \cap M_\Sigma$, then the multiplicity can equivalently be described as the degree of the map $\gamma \rightarrow S^1$ defined in (1.1).

Definition 1.4. Given a spinal open book π on M , a positive contact form α on M will be called a **Giroux form** for π if the following conditions hold:

- (1) The 2-form $d\alpha$ is positive on the interior of every page;
- (2) The Reeb vector field R_α is positively tangent to every oriented fiber of $\pi_\Sigma : M_\Sigma \rightarrow \Sigma$;
- (3) At ∂M , R_α is positively tangent to the fibers of $\pi_P|_{\partial M} : \partial M \rightarrow S^1$ and the characteristic foliation defined by $\ker \alpha$ on ∂M has only closed leaves, which are homologous on each connected component $T \subset \partial M$ to the preferred meridian m_T .³

A contact structure ξ on M will be said to be **supported** by π whenever it admits a contact form which is a Giroux form.

In order to obtain the existence and uniqueness of contact structures supported by a given spinal open book, technical issues will require us to examine the smooth compatibility of the spine and paper at their common boundary components slightly closer.

Definition 1.5. We will say that a spinal open book π admits a **smooth overlap** if the fibration $\pi_P : M_P \rightarrow S^1$ can be extended over an open neighborhood $M'_P \subset M$ containing M_P such that all fibers of π_Σ intersecting M'_P are contained in fibers of the extended π_P .

Remark 1.6. Any spinal open book can be modified, via a pair of smooth isotopies on the spine and paper which match on their common boundary components, so as to produce a spinal open book admitting a smooth overlap. The result of this “smoothing” operation is also unique up to isotopy.

In §2.3 we shall prove the following generalization of the standard theorem of Thurston and Winkelnkemper [TW75] on open books:

³The characteristic foliation $\ker(\alpha|_{T(\partial M)}) \subset T(\partial M)$ is oriented by any vector field X that satisfies $\Omega(X, \cdot) = \alpha|_{T(\partial M)}$ for a positive area form Ω on ∂M .

Theorem 1.7. *Suppose M is a compact oriented 3-manifold, possibly with boundary, and π is a spinal open book on M which admits a smooth overlap. Then the space of Giroux forms for π is nonempty and contractible. In particular, any isotopy class of spinal open books gives rise to a canonical isotopy class of supported contact structures.*

Remark 1.8. When $\partial M \neq \emptyset$, the above statement about uniqueness up to isotopy depends on the following version of Gray's stability theorem for manifolds with boundary: a smooth 1-parameter family of contact structures on a compact manifold with boundary is induced by a smooth isotopy if and only if the resulting characteristic foliations at the boundary are all isotopic. This follows by a variation on the usual proof of the standard version (see e.g. [Gei08]): if the characteristic foliations are isotopic, then after an isotopy near the boundary one can assume they are constant, and then check that the contact isotopy constructed in the standard way is generated by a vector field tangent to the boundary. For this reason it is important that supported contact structures always induce characteristic foliations on ∂M with closed leaves in a fixed homology class.

Example 1.9. An ordinary open book is the special case of a spinal open book where the spine is a tubular neighborhood of a transverse link $B \subset M$, i.e. each connected component of M_Σ is of the form $\mathbb{D}^2 \times S^1$, and the multiplicities of Definition 1.3 are all 1. Our definition of a Giroux form in this case does not quite match the standard one, but a Giroux form in our sense can be perturbed to the standard version, so the notion of a supported contact structure is the same.

Example 1.10. In the previous example, relaxing the condition that all multiplicities equal 1 generalizes from open books to certain types of *rational* open books as in [BEV12].

Example 1.11. Any *blown up summed open book* as defined in [Wen13a] can be viewed as a spinal open book whose vertebrae are all disks or annuli. For instance, one can understand the *binding sum* construction of [Wen13a] as follows. Topologically, it is defined by taking an ordinary open book $\pi : M \setminus B \rightarrow S^1$ with at least two binding circles $B_1, B_2 \subset B$, removing tubular neighborhoods of B_1 and B_2 and attaching the resulting boundary tori by an orientation reversing diffeomorphism that maps oriented boundaries of pages to each other and maps meridians to meridians (with reversed orientation). In terms of spinal open books, this is the same as removing two solid torus components

$$(\mathbb{D}^2 \times S^1) \amalg (\mathbb{D}^2 \times S^1) \subset M_\Sigma$$

from the spine and replacing these with $([-1, 1] \times S^1) \times S^1$, which we view as a spinal component with the annulus as a vertebra. In contact geometric terms, the binding sum on a supported contact structure produces a contact fiber sum (cf. [Gei08]), and it is not hard to show that the resulting contact structure is supported by the spinal open book described above.

Example 1.12. Spinal open books with boundary can always be constructed from closed spinal open books by deleting components of the spine and then choosing suitable preferred meridians. For example, suppose π is an ordinary open book as characterized in Example 1.9, so it is a spinal open book whose spinal components are all trivial fibrations $\mathbb{D}^2 \times S^1 \rightarrow \mathbb{D}^2$. We can then define a new spinal open book by deleting one such component $\mathbb{D}^2 \times S^1$ from the spine; this produces a new boundary component on the paper, which inherits a canonical meridian, namely $[\partial \mathbb{D}^2 \times \{*\}] \in H_1(\partial(\mathbb{D}^2 \times S^1))$. Topologically this has the effect of removing a tubular

neighborhood of one binding component, and the effect on supported contact structures is exactly what is described in [Wen13a] as *blowing up* along the binding.

Remark 1.13. Many of the notions of this section are also well defined without assuming that M is a globally smooth manifold: to define a spinal open book, M must at minimum be a topological manifold that is obtained by gluing together two smooth manifolds M_Σ and M_P along a smooth embedding $\partial M_\Sigma \hookrightarrow \partial M_P$, so there are well-defined smooth structures on M_Σ , M_P and $M_\Sigma \cap M_P$ but not necessarily on a neighborhood of the latter in M . In particular, it will be useful in the next section to take $M = \partial E$ where E is a smooth 4-manifold with boundary and corners; here the smooth faces of the boundary are M_Σ and M_P and the corner is $M_\Sigma \cap M_P$. In this case, a Giroux form will be assumed to be the restriction to $M = \partial E$ of a smooth 1-form on a neighborhood of the boundary in E , such that the conditions of Definition 1.4 are satisfied separately on each of the smooth faces M_Σ and M_P .

1.1.3. *Bordered Lefschetz fibrations.* The motivating example of a spinal open book is obtained by considering boundaries of Lefschetz fibrations. In the following, we assume E to be a smooth, compact, oriented and connected 4-manifold with boundary and corners such that ∂E is the union of two smooth faces

$$\partial E = \partial_h E \cup \partial_v E$$

which intersect at a corner of codimension two. Likewise, Σ will denote a compact, oriented and connected surface with nonempty boundary.

Definition 1.14. A **bordered Lefschetz fibration** of E over Σ is a smooth map $\Pi : E \rightarrow \Sigma$ with finitely many interior critical points $E^{\text{crit}} \subset \mathring{E}$ and critical values $\Sigma^{\text{crit}} \subset \mathring{\Sigma}$ such that the following conditions hold:

- (1) $\Pi^{-1}(\partial \Sigma) = \partial_v E$ and $\Pi|_{\partial_v E} : \partial_v E \rightarrow \partial \Sigma$ is a smooth fiber bundle;
- (2) $\Pi|_{\partial_h E} : \partial_h E \rightarrow \Sigma$ is also a smooth fiber bundle;
- (3) There exist integrable complex structures near E^{crit} and Σ^{crit} such that Π is holomorphic near E^{crit} and the critical points are nondegenerate;
- (4) All fibers $E_z := \Pi^{-1}(z)$ for $z \in \Sigma$ are connected and have nonempty boundary in $\partial_h E$.

We call E_z a **regular fiber** if $z \in \Sigma \setminus \Sigma^{\text{crit}}$ and otherwise a **singular fiber**; the latter are necessarily unions of smoothly immersed connected surfaces (the **irreducible components**) with positive transverse intersections. We say that Π is **allowable** if all the irreducible components of its fibers have nonempty boundary.

By the complex Morse lemma, one can find holomorphic coordinates near E^{crit} and Σ^{crit} so that Π takes the form

$$\Pi(z_1, z_2) = z_1^2 + z_2^2$$

near each critical point. Note also that in the standard language of vanishing cycles (cf. [GS99]), the “allowability” condition defined above is equivalent to requiring that no vanishing cycles be homologically trivial in the fiber.⁴

A bordered Lefschetz fibration $\Pi : E \rightarrow \Sigma$ naturally gives rise to a spinal open book on ∂E , with spine $M_\Sigma := \partial_h E$ and paper $M_P := \partial_v E$. The fibration $\pi_P : \partial_v E \rightarrow S^1$ is defined

⁴In some sources in the literature, it is erroneously stated that a Lefschetz fibration is allowable if and only if its vanishing cycles are always nonseparating in the fiber. We will often want to consider situations in which vanishing cycles are homologically nontrivial but separating, e.g. when the fiber is an annulus. In the case where fibers have genus zero, a Lefschetz fibration is allowable if and only if it is *relatively minimal*.

as the restriction $\Pi|_{\partial_v E} : \partial_v E \rightarrow \partial\Sigma$ after choosing an orientation preserving identification of each connected component of $\partial\Sigma$ with S^1 . Likewise, $\Pi|_{\partial_h E} : \partial_h E \rightarrow \Sigma$ defines a smooth fibration whose fibers are disjoint unions of finitely many circles, hence it can be factored as

$$\partial_h E \xrightarrow{\pi_\Sigma} \tilde{\Sigma} \xrightarrow{p} \Sigma,$$

where $\pi_\Sigma : \partial_h E \rightarrow \tilde{\Sigma}$ is a fiber bundle with connected fibers over another compact oriented surface $\tilde{\Sigma}$ with boundary, and $p : \tilde{\Sigma} \rightarrow \Sigma$ is a smooth finite covering map. As discussed in Remark 1.13, the fact that ∂E is not naturally a smooth manifold does not present any problem here. In this class of examples, every vertebra is a finite cover of the base Σ , the pages are all diffeomorphic to the regular fibers of Π , and the boundary components of these fibers form the fibers on the spine. Whenever π is a spinal open book on a 3-manifold M admitting a homeomorphism to ∂E that restricts to diffeomorphisms $M_\Sigma \rightarrow \partial_h E$ and $M_P \rightarrow \partial_v E$ such that π is related to $\Pi : E \rightarrow \Sigma$ as described above, we shall indicate this relationship by writing

$$\partial\Pi \cong \pi.$$

Clearly not all spinal open books can be obtained as boundaries of Lefschetz fibrations, so those that can deserve a special name.

Definition 1.15. A spinal open book π on a 3-manifold M will be called **symmetric** if

- (i) $\partial M = \emptyset$;
- (ii) All pages are diffeomorphic;
- (iii) For each of the vertebrae $\Sigma_1, \dots, \Sigma_r \subset \Sigma$, there are corresponding numbers $k_1, \dots, k_r \in \mathbb{N}$ such that every page has exactly k_i boundary components in $\pi_\Sigma^{-1}(\partial\Sigma_i)$ for $i = 1, \dots, r$.

We shall say that π is **uniform** if, in addition to the above conditions, there exists a fixed compact oriented surface Σ_0 whose boundary components correspond bijectively with the connected components of M_P such that for each $i = 1, \dots, r$ there exists a k_i -fold branched cover

$$\Sigma_i \rightarrow \Sigma_0$$

for which the restriction to each connected boundary component $\gamma \subset \partial\Sigma_i$ is an m_γ -fold cover of the component of $\partial\Sigma_0$ corresponding to the component of M_P touching $\pi_\Sigma^{-1}(\gamma)$, where m_γ denotes the multiplicity of π_P at $\pi_\Sigma^{-1}(\gamma)$ (see Definition 1.3). Finally, π is **Lefschetz-amenable** if it is uniform and all branched covers satisfying the above conditions have no branch points.

Remark 1.16. In many examples of interest—in particular for the circle bundles over oriented surfaces studied in §1.4 and further in [LVW], π is symmetric with $k_1 = \dots = k_r = 1$, in which case it is uniform if and only if all vertebrae are diffeomorphic. The Lefschetz-amenability condition is trivially satisfied in such cases since branched covers of degree 1 are diffeomorphisms. In more general situations, the uniformity and amenability conditions can often both be checked via the Riemann-Hurwitz formula;⁵ we will use this in [LVW] to classify the fillings of certain non-orientable contact circle bundles over non-orientable surfaces.

⁵Note that by capping Σ_i and Σ_0 with disks, the existence of the required branched cover $\Sigma_i \rightarrow \Sigma_0$ is equivalent to a question about the existence of a branched cover of closed surfaces with certain prescribed branching orders. Questions of this type can be subtle in general, but are trivial e.g. if the degree is 2, or more generally if all branch points are required to be simple, cf. [EKS84, Prop. 2.8].

The discussion above shows that for any bordered Lefschetz fibration $\Pi : E \rightarrow \Sigma$, the spinal open book $\pi := \partial\Pi$ is necessarily uniform, and the associated branched covers $\Sigma_i \rightarrow \Sigma$ have no branch points. The more precise version of Theorem A proved in [LVW] will imply that every spinal open book which contains a planar page and supports a strongly fillable contact structure must be uniform—moreover, if it is also amenable, then its strong fillings can be classified entirely in terms of Lefschetz fibrations.

Example 1.17. For any bordered Lefschetz fibration over the disk, the spinal open book induced at its boundary is an ordinary open book (see Example 1.9). In fact, any ordinary open book on a closed and connected 3-manifold, when regarded as a spinal open book, is uniform and Lefschetz-amenable. Of course not every open book is the boundary of a bordered Lefschetz fibration; this depends on its monodromy!

Example 1.18. For a bordered Lefschetz fibration over the annulus, if the fibration restricted to the horizontal boundary is trivial, then the induced spinal open book at the boundary is equivalent to a *symmetric summed open book* as defined in [Wen13a].

We now define various types of symplectic structures that are natural to consider on the total space of a bordered Lefschetz fibration $\Pi : E \rightarrow \Sigma$. Note that the orientations of E and Σ give rise to a natural orientation of the fibers. We shall say that a symplectic form ω on E is **supported** by Π whenever the following conditions hold:

- (1) Every oriented fiber is a symplectic submanifold away from E^{crit} ;
- (2) A neighborhood of E^{crit} admits a smooth almost complex structure J which restricts to a positively oriented complex structure on the smooth part of each fiber and satisfies $\omega(v, Jv) > 0$ for every nonzero vector $v \in TE|_{E^{\text{crit}}}$, i.e. J is *tamed* by ω at E^{crit} .

For the following definitions, assume always that ω is a symplectic structure supported by Π .

Definition 1.19. We say that ω is **weakly convex** if it can be written near $\partial_h E$ as $\omega = d\lambda$, where λ is a smooth 1-form that restricts to $\partial_h E$ as a contact form whose Reeb orbits are boundary components of fibers.

Definition 1.20. We say that ω is **strongly convex** if it can be written near ∂E as $\omega = d\lambda$, where λ is a smooth 1-form that restricts to ∂E as a Giroux form for $\pi = \partial\Pi$ (see Remark 1.13).

Definition 1.21. We say that ω is **Liouville** if it is strongly convex and the primitive λ of Definition 1.20 extends to a global primitive of ω on E .

These three definitions are designed so that a suitable smoothing of E at the corners will inherit the structure of a weak/strong/exact symplectic filling of (M, ξ) , with ξ supported by π , see §2.5.

To move from the Liouville to the Stein case, it will be convenient to introduce a notion that is intermediate between Weinstein and Stein structures.

Definition 1.22. Suppose W is a compact manifold with boundary, possibly also with corners. An **almost Stein structure** on W is a pair (J, f) consisting of an almost complex structure J and a smooth function $f : W \rightarrow \mathbb{R}$ such that, writing $\lambda := -df \circ J$, $d\lambda$ is a symplectic form taming J (i.e. f is *J -convex*) and λ restricts to a contact form on every smooth face of ∂W . If $M := \partial W$ is smooth and $\xi = \ker(\lambda|_{TM})$, we will call (W, J, f) an **almost Stein filling** of (M, ξ) .

We assign the natural C^∞ -topology to the space of almost Stein structures and say that two such structures are **almost Stein homotopic** if they lie in the same connected component of this space. Any Stein structure J determines an almost Stein structure (J, f) uniquely up to homotopy, where uniqueness follows from the fact that the space of exhausting J -convex functions is convex. We should point out two aspects of almost Stein structures that differ from Stein structures: first, J is not assumed integrable, and second, f is not assumed constant at the boundary (indeed, it *cannot* be constant on ∂W if there are corners). The latter has the consequence that for a fixed J , the space of functions f making (J, f) an almost Stein structure is *not* generally convex—linear interpolations between two such functions may fail to induce contact structures at the boundary. For this reason we can no longer regard the J -convex function as auxiliary data. It is clear however that any almost Stein structure (J, f) determines a Weinstein structure on the smooth manifold with boundary obtained by rounding the corners of ∂W , and this structure is canonical up to Weinstein homotopy. Indeed, the Liouville form $-df \circ J$ is dual to a Liouville vector field that points transversely outward at every smooth face of ∂W , and this vector field is automatically gradient-like with respect to f . One can therefore perturb f if necessary to make it Morse (cf. Remark 1.1), and then modify it outside a neighborhood of its critical points to a Lyapunov function that is constant on the smoothed boundary, the result being unique up to homotopy through Lyapunov functions fixed near the critical points. Using [CE12], this implies that for any manifold W with boundary and corners, there is a canonical one-to-one correspondence between almost Stein homotopy classes on W and Stein homotopy classes on W after smoothing corners.

Definition 1.23. Given a bordered Lefschetz fibration $\Pi : E \rightarrow \Sigma$, we will say that an almost Stein structure (J, f) on E is **supported by** Π if the following conditions are satisfied:

- There exists a complex structure j on Σ such that $\Pi : (E, J) \rightarrow (\Sigma, j)$ is pseudoholomorphic;
- The 1-form $\lambda_J := -df \circ J$ restricts to ∂E as a Giroux form for $\pi = \partial\Pi$ (see Remark 1.13);
- For every $z \in \Sigma$, f is constant on each connected component of ∂E_z ;
- The maximal J -complex subbundle of $T(\partial_h E)$ is invariant under the Reeb flow determined by $\lambda_J|_{T(\partial_h E)}$.

Observe that if (J, f) is supported by Π , then every fiber $E_z \subset E$ inherits a Stein structure $J|_{TE_z}$ with plurisubharmonic function $f|_{E_z}$, so in particular the fibers are both J -holomorphic and symplectic, and $-d(df \circ J)$ defines a supported Liouville structure.

The following variation on results of Thurston [Thu76] and Gompf [GS99] will be proved in §2.4.

Theorem 1.24. *For any 4-dimensional bordered Lefschetz fibration $\Pi : E \rightarrow \Sigma$, the spaces of supported symplectic structures that are weakly or strongly convex are both nonempty and contractible. Moreover, if the Lefschetz fibration is allowable, then the same is true for the spaces of supported Liouville structures and almost Stein structures. In each case, the corners can be smoothed to produce a weak/strong/exact/Stein filling of the contact manifold supported by $\pi := \partial\Pi$, and this filling is canonically defined up to deformation equivalence.*

1.2. Surgery on spinal open books. There are various natural topological operations on spinal open books that give rise to symplectic cobordisms. We now briefly describe two such operations.

1.2.1. *Spine removal surgery.* The following makes precise the non-exact cobordism construction that was sketched in Theorem D of the introduction, generalizing previous constructions from [Eli04, GS12, Wen13b] (see also the higher-dimensional analogues in [MNW13, DGZ14, Klu18]). For this discussion, it is useful to allow a slight loosening of the main definition of this paper: we will say that a **generalized spinal open book** is an object satisfying all the conditions of Definition 1.2 except that fibers of the paper $\pi_P : M_P \rightarrow S^1$ are allowed to have components with no boundary. A generalized spinal open book may therefore have some connected components that have neither spine nor boundary, but are simply fibrations of closed pages over S^1 ; note that a Giroux form cannot exist in this case, due to Stokes' theorem. Such an object is then a spinal open book in the usual sense—and thus supports a contact structure—if and only if it has no closed pages.

Suppose (M', ξ) is a closed contact 3-manifold containing a compact 3-dimensional submanifold M (possibly with boundary) on which ξ is supported by a spinal open book π with spine $\pi_\Sigma : M_\Sigma \rightarrow \Sigma$ and paper $\pi_P : M_P \rightarrow S^1$. Choose an open and closed subset

$$\Sigma^{\text{rem}} \subset \Sigma.$$

The boundary of the corresponding union of spinal components $\pi_\Sigma^{-1}(\Sigma^{\text{rem}}) \subset M_\Sigma$ is a disjoint union of 2-tori, each foliated by an S^1 -family of oriented circles which are fibers of π_Σ . We can then define a new compact manifold \widetilde{M} from M (and a closed manifold \widetilde{M}' from M') by removing the interior of $\pi_\Sigma^{-1}(\Sigma^{\text{rem}})$ and attaching solid tori $S^1 \times \mathbb{D}^2$ to each of the connected components of $\partial(\pi_\Sigma^{-1}(\Sigma^{\text{rem}}))$ so that the oriented circles $\{*\} \times \partial\mathbb{D}^2$ match the leaves of the foliation. (Some schematic pictures of this procedure are shown in Figures 10 and 11 in §4.2.) The new domain $\widetilde{M} \subset \widetilde{M}'$ inherits from π a generalized spinal open book $\widetilde{\pi}$ with spine $M_\Sigma \setminus \pi_\Sigma^{-1}(\Sigma^{\text{rem}})$ and pages that are obtained from the pages of π by attaching disks to cap every boundary component touching $\pi_\Sigma^{-1}(\Sigma^{\text{rem}})$. We say that $\widetilde{\pi}$ is obtained from π by **spine removal surgery**.

The spine removal operation corresponds to a cobordism that can be understood as a form of handle attachment. In particular, we can consider the compact 4-dimensional manifold with boundary and corners

$$X := ([0, 1] \times M') \cup_{\{1\} \times \pi_\Sigma^{-1}(\Sigma^{\text{rem}})} (\Sigma^{\text{rem}} \times \mathbb{D}^2),$$

where $\Sigma^{\text{rem}} \times \partial\mathbb{D}^2$ is identified with $\pi_\Sigma^{-1}(\Sigma^{\text{rem}})$ via a choice of trivialization $\pi_\Sigma^{-1}(\Sigma^{\text{rem}}) \cong \Sigma^{\text{rem}} \times S^1$. After smoothing the corners, we have

$$\partial X = -M' \amalg \widetilde{M}'.$$

The following result will be proved in §4.2.

Theorem 1.25. *Suppose Ω is a closed 2-form on M' such that $\Omega|_\xi > 0$ and $\Omega|_{\pi_\Sigma^{-1}(\Sigma^{\text{rem}})}$ is exact. Then for any choice of compact subset Σ_0 in the interior of Σ^{rem} , the cobordism X described above admits a symplectic structure ω with the following properties:*

- (1) $\omega|_{TM'} = \Omega$.
- (2) ω is positive on the interior of every page of $\widetilde{\pi}$.
- (3) On every connected component of \widetilde{M}' that is not foliated by closed pages of $\widetilde{\pi}$, there exists a contact structure $\widetilde{\xi}$ which is supported by $\widetilde{\pi}$ in \widetilde{M} , matches ξ on $\widetilde{M}' \setminus \widetilde{M} = M' \setminus M$, and satisfies $\omega|_{\widetilde{\xi}} > 0$.

- (4) For every $z \in \Sigma_0$, the core $\{z\} \times \mathbb{D}^2$ and co-core $\Sigma^{\text{rem}} \times \{0\}$ of the handle $\Sigma^{\text{rem}} \times \mathbb{D}^2$ are both symplectic submanifolds, the former with reversed orientation.

We will see in the proof that the disk \mathbb{D}^2 in the symplectic handle $\Sigma^{\text{rem}} \times \mathbb{D}^2$ can freely be replaced by any other compact oriented surface with connected boundary; more generally, one could equally well remove several spine components at once and replace them with $\Sigma^{\text{rem}} \times S$ for a compact oriented surface S with the right number of boundary components. The key intuition is to view S as a symplectic *cap* for the appropriate disjoint union of fibers in the contact circle fibration $\pi_\Sigma : M_\Sigma \rightarrow \Sigma$, and this is also the right perspective in higher-dimensional cases such as [DGZ14, Klu18]. We will not comment any further on these generalizations since the applications in this paper do not require them.

1.2.2. *Fiber connected sum along pages.* The following is a special case of a construction due to R. Avdek [Avd21]. As in the previous section, suppose (M', ξ) is a closed contact 3-manifold containing a compact domain $M \subset M'$ on which ξ is supported by a spinal open book π . Suppose S is a compact, connected and oriented surface with boundary, and $S_0, S_1 \subset M_P$ are pages of π admitting orientation preserving diffeomorphisms

$$\psi_i : S \rightarrow S_i, \quad i = 0, 1.$$

By a minor adjustment to the proof of Theorem 1.7 (see Lemma 2.7 in particular), one can find a Giroux form α for π on M such that $\psi_0^* \alpha = \psi_1^* \alpha$. In the terminology of [Avd21], S_0 and S_1 can then be regarded as a pair of identical *Liouville hypersurfaces* in M . Choose neighborhoods $[-1, 1] \times S_i \cong \mathcal{N}(S_i) \subset M_P$ of S_i for $i = 0, 1$ and define the compact 4-manifold with boundary and corners

$$X := ([0, 1] \times M') \cup_{\mathcal{N}(S_0) \sqcup \mathcal{N}(S_1)} ([0, 1] \times [-1, 1] \times S),$$

by identifying $\{i\} \times [-1, 1] \times S$ with $\mathcal{N}(S_i)$ for $i = 0, 1$. After smoothing corners, we have

$$\partial X = -M' \sqcup \widetilde{M}',$$

where \widetilde{M}' is obtained from M' by performing a so-called *Liouville connected sum* along S_0 and S_1 . Then \widetilde{M}' contains a compact subdomain \widetilde{M} which naturally carries a spinal open book $\widetilde{\pi}$; it is obtained from π by attaching 1-handles to the vertebrae and concatenating families of pages correspondingly. The following is an immediate consequence of the main result in [Avd21]:

Theorem. *The manifold X described above can be given the structure of a Stein cobordism with concave boundary (M', ξ) and convex boundary $(\widetilde{M}', \widetilde{\xi})$, where $\widetilde{\xi}$ is a contact structure which matches ξ on $\widetilde{M}' \setminus \widetilde{M} = M' \setminus M$ and is supported by $\widetilde{\pi}$ on \widetilde{M} .*

It was observed in [Avd21] that the simplest case of this operation turns ordinary open books into *symmetric summed open books* in the sense of [Wen13a], i.e. disk vertebrae become annuli. More generally, this construction can be used to give an alternative proof of the fact that allowable bordered Lefschetz fibrations over arbitrary compact oriented surfaces always admit Stein structures—the details of this argument have been worked out by Baykur and the second author, see [BV15].

1.3. Partially planar domains, torsion and filling obstructions. We now state a few theorems that are straightforward generalizations of results from [Wen13a], and will all be proved in §5 using spine removal surgery. Most of them can also be derived from algebraic counterparts that we will prove in [LVW], involving contact invariants in symplectic field theory and embedded contact homology.

The following is the basic condition needed in order to apply the machinery of pseudoholomorphic curves in studying spinal open books.

Definition 1.26. A 3-dimensional spinal open book will be called **partially planar** if its interior contains a page of genus zero. A compact contact 3-manifold (M, ξ) , possibly with boundary, will be called a **partially planar domain** if ξ is supported by a partially planar spinal open book. We then refer to any interior connected component of the paper containing planar pages as a **planar piece**.

Definition 1.27. Suppose (M, ξ) is a closed contact 3-manifold and Ω is a closed 2-form on M . A partially planar domain M_0 embedded in (M, ξ) is called **Ω -separating** if it has a planar piece $M_0^P \subset \overset{\circ}{M}_0$ such that Ω is exact on every spinal component touching M_0^P . It is called **fully separating** if this is true for all closed 2-forms Ω on M .

Note that this condition depends only on the cohomology class $[\Omega] \in H_{\text{dR}}^2(M)$, and it is vacuous if Ω is exact. We will see that it determines precisely which results on the strong fillings of spinal open books admit extensions for *weak* fillings.

Example 1.28. Since all closed 2-forms are exact on a solid torus $\mathbb{D}^2 \times S^1$, every planar open book is a fully separating partially planar domain (cf. Example 1.9). As explained in [Wen13a, Wen10], a Giroux torsion domain can also be viewed as a partially planar domain in terms of the binding sum construction, but its spinal components are thickened 2-tori and thus can have cohomology, so such a domain is fully separating if and only if it separates the ambient 3-manifold.

Our first main result about partially planar domains generalizes the main theorem from [ABW10]; indeed, taking $\Omega = 0$ in the following statement produces an obstruction to the existence of non-separating hypersurfaces of contact type.

Theorem 1.29. *Suppose (M, ξ) is a closed contact 3-manifold, Ω is a closed 2-form on M and (M, ξ) contains an Ω -separating partially planar domain. Then there exists no closed symplectic 4-manifold (W, ω) admitting a non-separating embedding $\iota : M \hookrightarrow W$ for which $\iota^*\omega|_{\xi} > 0$ and $[\iota^*\omega] = [\Omega] \in H_{\text{dR}}^2(M)$.*

The following related result generalizes a planarity obstruction originally due to Etnyre [Etn04b]. Recall that (W, ω) is called a symplectic **semifilling** of (M, ξ) whenever it is a filling of the disjoint union of (M, ξ) with some other (possibly empty) contact manifold.

Corollary 1.30. *If (M, ξ) is a closed contact 3-manifold containing a partially planar domain, then it admits no weak semifilling (W, ω) with disconnected boundary for which the partially planar domain is $(\omega|_{TM})$ -separating.*

Proof. We use a suggestion by Etnyre that first appeared in [ABW10]: if such a semifilling exists, then one can attach a Weinstein 1-handle to build a weak filling of the boundary connected sum of its two components, and then cap the result via [Eli04] or [Etn04a]. This produces a closed symplectic manifold (W, ω) that contains (M, ξ) as a non-separating hypersurface in violation of Theorem 1.29. \square

Next, we can consider the natural generalization of the local filling obstruction known as *planar k -torsion* from [Wen13a] into the spinal open book setting.

Definition 1.31. Suppose (M, ξ) is a closed contact 3-manifold and Ω is a closed 2-form on M . Then for $k \geq 0$ an integer, a partially planar domain $M_0 \subset M$ is called a **(spinal) planar torsion domain of order k** (or simply a *planar k -torsion domain*) if it is not symmetric and contains an interior planar piece $M_0^P \subset \overset{\circ}{M}_0$ whose pages have $k + 1$ boundary components. Further, it is an **Ω -separating** planar k -torsion domain if Ω is exact on all spinal components touching M_0^P , and a **fully separating** planar k -torsion domain if this is true for all closed 2-forms Ω on M . Any contact 3-manifold containing such a domain is said to have (perhaps **Ω -separating** or **fully separating**) **planar k -torsion**.

The less general version of this definition in [Wen13a] was expressed in the framework of *blown up summed open books*, i.e. spinal open books whose vertebrae are all disks and annuli. We have inserted the word “spinal” in front of “planar torsion” in the above definition to distinguish the new notion from the less general version, but we shall usually drop the word “spinal” from the nomenclature: this will not cause any confusion since anything satisfying the old definition also satisfies the new one, and the results we are able to prove with the new definition parallel results in [Wen13a] almost exactly. For instance, the less general version of planar 0-torsion was shown in [Wen13a] to be equivalent to overtwistedness, and this is still true in the new framework:

Proposition 1.32. *A closed contact 3-manifold is overtwisted if and only if it has planar 0-torsion.*

Proof. If (M, ξ) is overtwisted, then Eliashberg’s flexibility result [Eli89] implies that (M, ξ) contains a so-called *Lutz tube*, and any neighborhood of this contains a planar 0-torsion domain by [Wen13a, Prop. 2.19]. For the converse, see Lemma 2.9. \square

It was also shown in [Wen13a] that anything with Giroux torsion also has planar 1-torsion, but it was left open whether the converse might be true. Working with spinal open books makes it easy to find a counterexample to this converse:

Proposition 1.33. *If (M, ξ) is a closed contact 3-manifold with positive Giroux torsion then it has planar 1-torsion. However, there exist closed contact 3-manifolds that have planar 1-torsion but no Giroux torsion.*

Proof. The fact that Giroux torsion implies 1-torsion was shown in [Wen13a]; in fact, any Giroux torsion domain has an open neighborhood that contains a planar 1-torsion domain whose pages and vertebrae are all annuli. Some examples with planar 1-torsion but no Giroux torsion are exhibited in §1.4; see Corollary 1.39 and Remark 1.40. \square

We will prove the following statement in §5 by using spine removal surgery to reduce it to standard results in closed holomorphic curve theory.

Theorem 1.34. *If (M, ξ) has planar torsion, then it is not strongly fillable. Moreover, if (M, ξ) has Ω -separating planar torsion for some closed 2-form Ω on M , then it admits no weak filling (W, ω) with $\omega|_{TM}$ cohomologous to M . In particular (M, ξ) is not weakly fillable whenever it has fully separating planar torsion.*

1.4. Fillability of circle bundles. As an application of the filling obstructions in the previous subsection, we now exhibit a large class of non-fillable contact 3-manifolds that were not previously accessible to holomorphic curve methods. (Some of them can be understood using techniques from Heegaard Floer homology; see especially [HKM, Mas12].) They take the form of circle bundles with S^1 -invariant contact structures partitioned by multicurves.

Throughout this subsection, assume $\pi : M \rightarrow B$ is a smooth S^1 -bundle with structure group $O(2)$ acting on the circle by rotations and reflections, where the base B is a closed and connected (but not necessarily orientable) surface, and the total space M is oriented. If B is orientable, then the $O(2)$ -structure lifts to the structure of a principal S^1 -bundle, with the S^1 -action defined up to a sign, so we can speak of S^1 -invariant contact structures on M . More generally, we will abuse terminology and call a contact structure S^1 -**invariant** if its expression in every $O(2)$ -compatible local trivialization of $\pi : M \rightarrow B$ is $O(2)$ -invariant. This is the same as saying that it lifts to an S^1 -invariant contact structure on the induced fibration over the canonical oriented double cover of B . As usual, all contact structures in this discussion are assumed to be positive and co-oriented.

Any S^1 -invariant contact structure ξ on M determines a 1-dimensional submanifold $\Gamma \subset B$ (i.e. a **multicurve**) consisting of all points at which the fiber is Legendrian. One says in this case that ξ is **partitioned by** Γ . Notice that outside of Γ , transversality to ξ determines an orientation of the bundle and therefore an orientation of $B \setminus \Gamma$. Moreover, Γ automatically has the following property:

Definition 1.35. Suppose B is a closed surface and $\Gamma \subset B$ is a multicurve such that $B \setminus \Gamma$ is oriented. We say that Γ **inverts orientations** if for every sufficiently small neighborhood $\mathcal{U} \subset B$ that is divided by Γ into two components \mathcal{U}_+ and \mathcal{U}_- , \mathcal{U} can be given an orientation that matches that of $B \setminus \Gamma$ on \mathcal{U}_+ and is the opposite on \mathcal{U}_- .

If B is orientable, this condition simply means that Γ divides B into components B_+ and B_- (each possibly disconnected) which inherit opposite orientations. Some concrete examples where B is non-orientable (in particular the Klein bottle) can be constructed in the form of contact parabolic torus bundles; see [LVW]. The following result is due to Lutz [Lut77] in the orientable case, and in general it can easily be derived from Theorem 1.7 via Proposition 1.37 below.

Proposition 1.36. *Suppose $\pi : M \rightarrow B$ is a smooth circle bundle with structure group $O(2)$, where B is a closed connected surface and M is oriented, and $\Gamma \subset B$ is a nonempty multicurve such that $B \setminus \Gamma$ is orientable and Γ inverts orientations. Then each choice of orientation on $B \setminus \Gamma$ determines an S^1 -invariant contact structure ξ_Γ that is partitioned by Γ and is positively transverse to the fibers over $B \setminus \Gamma$. Moreover, the contact structure with these properties is unique up to isotopy.*

We will see in [LVW] that the strong symplectic fillings of each circle bundle (M, ξ_Γ) arising from the above proposition can be classified completely whenever its base is orientable, and also in some cases where the base is not orientable. The basic observation behind this is that there is a natural correspondence between S^1 -bundles $\pi : M \rightarrow B$ with nonempty multicurves $\Gamma \subset B$ satisfying the stated conditions in Prop. 1.36 and spinal open book decompositions of M with annular pages. Topologically this is easy to see: choosing a tubular neighborhood $\mathcal{U}_\Gamma \subset B$ of Γ , we identify each connected component of the closure $\overline{\mathcal{U}}_\Gamma$ with an interval bundle over S^1 , which gives $\pi^{-1}(\overline{\mathcal{U}}_\Gamma)$ the structure of a disjoint union of smooth annulus bundles over S^1 whose fibers each have boundary equal to some pair of oriented fibers of π . We

therefore call $\pi^{-1}(\overline{\mathcal{U}}_\Gamma)$ with its associated fibration over S^1 the paper $\pi_P : M_P \rightarrow S^1$, and the spine $\pi_\Sigma : M_\Sigma \rightarrow \Sigma$ is defined as the restriction of π to $\pi^{-1}(B \setminus \mathcal{U}_\Gamma)$, with the fibers oriented to be compatible with the orientation of $\Sigma := B \setminus \mathcal{U}_\Gamma \subset B \setminus \Gamma$. Note that if B is orientable, then every component of $\Gamma \subset B$ has trivial normal bundle, so the monodromies of components of $\pi_P : M_P \rightarrow S^1$ (defined after choosing a trivialization of $\pi_\Sigma : M_\Sigma \rightarrow \Sigma$) can be taken to be powers of Dehn twists, fixing the boundary of the annulus. This is not always true if B is non-orientable: in particular, if $\gamma \subset \Gamma$ is a component whose neighborhood in B is a Möbius band, then the monodromy of the corresponding component of $\pi_P : M_P \rightarrow S^1$ interchanges the boundary components of the annulus, meaning this component of M_P has connected boundary, with multiplicity 2 (cf. Definition 1.3).

Proposition 1.37. *The spinal open book π associated to a circle bundle $\pi : M \rightarrow B$ and nonempty multicurve $\Gamma \subset B$ as described above supports a contact structure that is S^1 -invariant and partitioned by Γ . Moreover, any S^1 -invariant contact structure partitioned by Γ is isotopic to one that is supported by π .*

Proof. It is easy to check that the supported contact structure constructed by Theorem 1.7 in this setting is S^1 -invariant and partitioned by Γ . In the other direction, suppose ξ_Γ is S^1 -invariant and partitioned by Γ , and choose a tubular neighborhood \mathcal{U}_Γ of $\Gamma \subset B$ such that the pages of the resulting fibration $\pi_P : M_P \rightarrow S^1$ are tangent to ξ_Γ along $\pi^{-1}(\Gamma)$. This means that every contact form for ξ_Γ has a Reeb vector field transverse to the pages in some neighborhood of $\pi^{-1}(\Gamma)$. On $\pi^{-1}(B \setminus \mathcal{U}_\Gamma)$, ξ_Γ is positively transverse to the contact vector field which generates a fiber-preserving S^1 -action, hence one can choose a contact form for which this vector field is the Reeb field. One can now piece this together with a contact form near $\pi^{-1}(\Gamma)$ whose Reeb field is transverse to the pages so that the conditions of a Giroux form are satisfied. \square

Since the pages of the natural spinal open book on (M, ξ_Γ) have genus zero, we can immediately apply the $\Omega = 0$ case of Theorem 1.29 and Corollary 1.30 to conclude:

Corollary 1.38. *For any nonempty multicurve $\Gamma \subset B$ as in Proposition 1.36, the resulting contact circle bundle (M, ξ_Γ) admits no non-separating contact-type embeddings into any closed symplectic 4-manifold, and it also admits no strong semifillings with disconnected boundary.* \square

It is similarly easy to identify cases in which (M, ξ_Γ) has planar torsion, and is therefore not strongly fillable. If there is torsion it will be of order 1, since the pages of the spinal open book π supporting (M, ξ_Γ) are annuli, but the key question is in which cases π is symmetric. The vertebrae of π are equivalent to the connected components B_1, \dots, B_r of $B \setminus \Gamma$, and the pages come in S^1 -families corresponding to the connected components of Γ . Given a component $\gamma \subset \Gamma$, if it bounds the component $B_j \subset B \setminus \Gamma$, then B_j lies either on one side of γ or on both, where the latter is possible only if γ has a non-orientable normal bundle, so B is non-orientable. Symmetry then means that there exist fixed numbers $k_1, \dots, k_r \in \{0, 1, 2\}$ such that for each $j = 1, \dots, r$, every component of Γ touches B_j on exactly k_j sides. Clearly none of the k_j can be 0 in this case, since it would mean there is a component B_j whose closure does not touch Γ at all. If any $k_j = 2$, then it means every component of Γ must have that particular component B_j on both sides, hence $r = 1$ and B is not orientable. In the remaining case, $k_1 = \dots = k_r = 1$, and since at most two components B_j can touch each component of Γ , we conclude $r = 2$ and B is orientable with Γ splitting it into two components. We've proved:

Corollary 1.39. *Suppose ξ_Γ is an S^1 -invariant contact structure on a circle bundle $\pi : M \rightarrow B$, partitioned by a nonempty multicurve Γ , and that either of the following holds:*

- (i) $B \setminus \Gamma$ has at least three connected components;
- (ii) $B \setminus \Gamma$ is disconnected and B is non-orientable.

Then (M, ξ_Γ) has (untwisted) planar 1-torsion, so in particular it is not strongly fillable. \square

Remark 1.40. If B is oriented with positive genus and M is not a torus bundle, then [Mas12, Theorem 3] implies that (M, ξ_Γ) has zero Giroux torsion whenever no two connected components of Γ are isotopic. Using the theorem in [LVW] that planar 1-torsion implies algebraic 1-torsion, one can now extract from Corollary 1.39 many new examples of contact manifolds with algebraic 1-torsion but no Giroux torsion. This generalizes a result for trivial circle bundles that was proved in [LW11].

Remark 1.41. Corollaries 1.38 and 1.39 do not generally hold for *weak* fillings or semifillings. Indeed, [NW11, Theorem 5] implies that whenever every connected component of the multicurve $\Gamma \subset B$ is nonseparating, (M, ξ_Γ) admits both a weak filling and a weak semifilling with disconnected boundary. We can conclude from Corollary 1.30 and Theorem 1.34 that the symplectic structures of these weak fillings must always be nonexact on some spinal component in (M, ξ_Γ) .

2. CONTACT AND SYMPLECTIC STRUCTURES

The main objectives of this section are the proofs of Theorems 1.7 and 1.24 about the existence and uniqueness of supported contact and symplectic structures, plus a related result about almost Stein structures that will be needed for the classification of Stein fillings in [LVW]. We begin in §2.1 with a short collection of “Thurston-type” lemmas for defining contact or symplectic structures on fibrations. Section 2.2 will then fix some notation for collar coordinates and open coverings of spinal open books that will be useful throughout the rest of the paper. Theorem 1.7 is proved in §2.3, and the proof of Theorem 1.24 is carried out mainly in §2.4, with the detail about smoothing corners dealt with in §2.5.

2.1. Several varieties of the Thurston trick. Since it will be useful in a wide range of contexts, we collect in this subsection several elementary results that are variations on the main trick behind Thurston’s construction of symplectic forms on total spaces of symplectic fibrations [Thu76]. All of these results have higher-dimensional analogues, but with the exception of Remark 2.3, we will keep things as brief as possible by focusing on dimensions 3 and 4.

All manifolds in the following will be compact and oriented, and though it will not yet play a serious role in the discussion, they may also have boundary or corners.

2.1.1. Contact forms.

Proposition 2.1. *Assume M is a compact oriented 3-manifold, $\pi : M \rightarrow S^1$ is a submersion, σ is a positively oriented volume form on S^1 , and λ is a 1-form on M such that $d\lambda$ is positive on every fiber of π . Then $\lambda_K := \lambda + K \pi^* \sigma$ is a contact form for all $K \gg 0$, and this is true for all $K \geq 0$ if λ is contact.*

Proof. We have $\pi^* \sigma \wedge d\lambda > 0$ since $d\lambda$ is positive on fibers, so the result follows by writing

$$\lambda_K \wedge d\lambda_K = K \left(\pi^* \sigma \wedge d\lambda + \frac{1}{K} \lambda \wedge d\lambda \right).$$

□

Proposition 2.2. *Assume M is a compact oriented 3-manifold, Σ is a compact oriented surface, $\pi : M \rightarrow \Sigma$ is a submersion, σ is a 1-form on Σ with $d\sigma > 0$, and λ is a 1-form on M that is positive on every fiber of π and satisfies $d\lambda(v, \cdot) = 0$ for all $v \in \ker T\pi$. Then $\lambda_K := \lambda + K\pi^*\sigma$ is a contact form for all $K \gg 0$, and this is true for all $K \geq 0$ if λ is contact.*

Proof. Since $\pi^*\sigma$ and $d\lambda$ both annihilate vertical vectors, we have $\pi^*\sigma \wedge d\lambda \equiv 0$, but also $\lambda \wedge \pi^*d\sigma > 0$ due to the condition $d\sigma > 0$ and the positivity of λ on fibers. The result thus follows by writing

$$\lambda_K \wedge d\lambda_K = K \left(\lambda \wedge \pi^*d\sigma + \frac{1}{K} \lambda \wedge d\lambda \right).$$

□

Remark 2.3. If we regard σ in Proposition 2.1 as a contact form on S^1 and the fibers of $\pi : M \rightarrow S^1$ as Liouville domains with respect to λ , then the result has a straightforward generalization to higher dimensions as a statement about a Liouville fibration over a contact manifold. Proposition 2.2 similarly becomes a statement about a contact fibration over a Liouville domain (Σ, σ) , the only subtle point being the condition that $d\lambda$ should annihilate vertical vectors: if $\pi : M \rightarrow \Sigma$ is a 3-dimensional fibration, then the secret meaning of this condition is that it reduces the structure group to the group of *strict* contactomorphisms on S^1 , i.e. diffeomorphisms that preserve a fixed contact form and not only a contact structure. The natural generalization to higher dimensions can thus be phrased in terms of strict contact fibrations.

2.1.2. Symplectic and Liouville forms.

Proposition 2.4. *Assume E is a compact oriented 4-manifold, Σ is a compact oriented surface, $\Pi : E \rightarrow \Sigma$ is a submersion, μ is a positive area form on Σ and ω is a 2-form on E that is positive on all fibers of Π . Then $\omega_K := \omega + K\Pi^*\mu$ is symplectic for all $K \gg 0$, and this is true for all $K \geq 0$ if ω is symplectic.*

Proof. The positivity of ω on fibers implies $\Pi^*\mu \wedge \omega > 0$, so the result follows by writing

$$\omega_K \wedge \omega_K = K \left(2\Pi^*\mu \wedge \omega + \frac{1}{K} \omega \wedge \omega \right).$$

□

If both Σ and the fibers of $\Pi : E \rightarrow \Sigma$ have nonempty boundary, then we can also state a version specially for exact symplectic forms; note that in this case E must be a manifold with boundary *and corners*.

Corollary 2.5. *Assume E is a compact oriented 4-manifold, Σ is a compact oriented surface, $\Pi : E \rightarrow \Sigma$ is a submersion, σ is a 1-form on Σ satisfying $d\sigma > 0$, and λ is a 1-form on E such that $d\lambda$ is positive on fibers of Π . Then $\lambda_K := \lambda + K\Pi^*\sigma$ is a Liouville form for all $K \gg 0$, and this is true for all $K \geq 0$ if λ is Liouville.* □

2.1.3. J -convex functions. Recall that on an almost complex manifold (W, J) , a smooth function $f : W \rightarrow \mathbb{R}$ is called **J -convex** (or **plurisubharmonic**) if the 1-form $\lambda_J := -df \circ J$ is the primitive of a symplectic form $d\lambda_J$ that tames J .

Proposition 2.6. *Assume (E, J) is a compact almost complex 4-manifold, (Σ, j) is a compact Riemann surface, $\Pi : (E, J) \rightarrow (\Sigma, j)$ is a pseudoholomorphic submersion, $\varphi : \Sigma \rightarrow \mathbb{R}$ is a j -convex function and $f : E \rightarrow \mathbb{R}$ is a function whose restriction to every fiber of Π is J -convex. Then $f_K := f + K(\varphi \circ \Pi)$ is a J -convex function for all $K \gg 0$, and this is true for all $K \geq 0$ if f is J -convex.*

Proof. The 1-forms $\sigma := -d\varphi \circ j$ on Σ and $\lambda = -df \circ J$ on E satisfy the hypotheses of Corollary 2.5, and since $T\Pi \circ J = j \circ T\Pi$, we have

$$\lambda_K := -df_K \circ J = -df \circ J - K d(\varphi \circ \Pi) \circ J = \lambda + K \Pi^* \sigma.$$

Then for any nontrivial $v \in TE$,

$$d\lambda_K(v, Jv) = K \Pi^* d\sigma(v, Jv) + d\lambda(v, Jv) = K \left[d\sigma(\Pi_* v, j\Pi_* v) + \frac{1}{K} d\lambda(v, Jv) \right].$$

This is clearly positive for all $K \geq 0$ if $d\lambda$ tames J ; more generally, the second term will always be positive when v lies in some neighborhood of the vertical subbundle, and if v is outside of this neighborhood, then this term will be dominated by $d\sigma(\Pi_* v, j\Pi_* v)$ as long as $K > 0$ is large enough. \square

2.2. Collar neighborhoods and coordinates. In this section we fix some notation that will be useful throughout the rest of the paper.

Fix a compact 3-manifold M with spinal open book

$$\pi = (\pi_\Sigma : M_\Sigma \rightarrow \Sigma, \pi_P : M_P \rightarrow S^1, \{m_T\}_{T \subset \partial M}),$$

and choose an oriented foliation \mathcal{F} of ∂M with closed leaves that represent the homology classes m_T . The circle bundle $\pi_\Sigma : M_\Sigma \rightarrow \Sigma$ is necessarily trivializable, so for convenience we shall fix an identification of M_Σ with $\Sigma \times S^1$ such that

$$\pi_\Sigma : M_\Sigma = \Sigma \times S^1 \rightarrow \Sigma : (z, \theta) \mapsto z.$$

This defines the coordinate $\theta \in S^1$ globally on M_Σ . The boundary $\partial\Sigma$ admits a collar neighborhood

$$\mathcal{N}(\partial\Sigma) \subset \Sigma$$

whose connected components can be identified with $(-1, 0] \times S^1$, carrying coordinates (s, ϕ) . We shall denote the resulting collar neighborhood of ∂M_Σ in M_Σ by

$$\mathcal{N}(\partial M_\Sigma) := \pi_\Sigma^{-1}(\mathcal{N}(\partial\Sigma)) = \mathcal{N}(\partial\Sigma) \times S^1;$$

its connected components are identified with $(-1, 0] \times S^1 \times S^1$ by our chosen trivialization and thus carry coordinates (s, ϕ, θ) .

The paper can be identified in turn with a mapping torus

$$M_P = (\mathbb{R} \times P) / \sim, \quad \text{where } (\tau, p) \sim (\tau + 1, \mu(p)),$$

$$M_P \xrightarrow{\pi_P} S^1 = \mathbb{R}/\mathbb{Z} : [(\tau, p)] \rightarrow [\tau],$$

where the fiber $P := \pi_P^{-1}(*)$ is a compact oriented (but not necessarily connected) surface with boundary, and the monodromy $\mu : P \rightarrow P$ is an orientation-preserving diffeomorphism that preserves each connected component of P . In contrast to the setting of ordinary open books,

here we must allow the possibility that μ is nontrivial near the boundary, e.g. it may permute boundary components. We can assume without loss of generality however that ∂P has a collar neighborhood $\mathcal{N}(\partial P) \subset P$ whose connected components have coordinates $(t, \theta) \in (-1, 0] \times S^1$ in which $\mu(t, \theta) = (t, \theta)$, hence the corresponding collar neighborhood

$$\mathcal{N}(\partial M_P) \subset M_P$$

of ∂M_P is identified with

$$(\mathbb{R} \times (-1, 0] \times S^1 \times \{1, \dots, N\}) / (\tau, t, \theta, i) \sim (\tau + 1, t, \theta, \sigma(i)) \subset M_P$$

for $N := \#\pi_0(\partial P)$ and some permutation $\sigma \in S_N$. The connected components of $\mathcal{N}(\partial M_P)$ are now in one-to-one correspondence with the invariant subsets of $\{1, \dots, N\}$ on which the cyclic subgroup $\langle \sigma \rangle \subset S_N$ generated by σ acts transitively. Given such a subset $I \subset \{1, \dots, N\}$, if the action of $\langle \sigma \rangle$ on I has order $m \in \mathbb{N}$, then the map $[(\tau, t, \theta, i)] \mapsto ([\tau/m], t, \theta)$ identifies the corresponding component of $\mathcal{N}(\partial M_P)$ diffeomorphically with $S^1 \times (-1, 0] \times S^1$. Denote the resulting coordinates on components of $\mathcal{N}(\partial M_P)$ by (ϕ, t, θ) . We now have

$$\pi_P(\phi, t, \theta) = m\phi,$$

where the integer $m \in \mathbb{N}$ may vary for different components of $\mathcal{N}(\partial M_P)$. These integers are the **multiplicities** of $\pi_P : M_P \rightarrow S^1$ at its boundary components (cf. Definition 1.3). Note that there is some freedom to change these coordinates on each component of $\mathcal{N}(\partial M_P)$ without changing the formula for π_P , thus we can assume without loss of generality that the chosen oriented foliation \mathcal{F} on ∂M with leaves homologous to the preferred meridians is generated by the flow of the coordinate vector field ∂_ϕ .

To summarize, we have defined collar neighborhoods of the boundary in Σ , M_Σ and M_P whose connected components carry positively oriented coordinates as follows:

$$\begin{aligned} (s, \phi) &\in (-1, 0] \times S^1 \subset (-1, 0] \times \partial\Sigma = \mathcal{N}(\partial\Sigma) \subset \Sigma, \\ (s, \phi, \theta) &\in (-1, 0] \times S^1 \times S^1 \subset (-1, 0] \times \partial M_\Sigma = \mathcal{N}(\partial M_\Sigma) \subset M_\Sigma, \\ (\phi, t, \theta) &\in S^1 \times (-1, 0] \times S^1 \subset (-1, 0] \times \partial M_P = \mathcal{N}(\partial M_P) \subset M_P. \end{aligned}$$

These coordinates satisfy $\pi_\Sigma(s, \phi, \theta) = (s, \phi) \in \mathcal{N}(\partial\Sigma)$ on $\mathcal{N}(\partial M_\Sigma)$ and $\pi_P(\phi, t, \theta) = m\phi \in S^1$ on $\mathcal{N}(\partial M_P)$, where the multiplicity $m \in \mathbb{N}$ may have different values on distinct connected components of $\mathcal{N}(\partial M_P)$. We can also assume without loss of generality that the coordinate labels are consistent in the sense that the induced 2-torus coordinates

$$(\phi, \theta) \in S^1 \times S^1 \subset \partial M_\Sigma$$

match the corresponding ϕ - and θ -coordinates defined on $\mathcal{N}(\partial M_P)$ wherever it overlaps $\mathcal{N}(\partial M_\Sigma)$.

To continue, let us add the assumption that π admits a smooth overlap (see Definition 1.5). We can then introduce a decomposition of M into open subsets

$$M = \widetilde{M}_\Sigma \cup \widetilde{M}_I \cup \widetilde{M}_P \cup \widetilde{M}_\partial$$

defined as follows:

- \widetilde{M}_P is the complement of $\{t \geq -1/2\} \subset \mathcal{N}(\partial M_P)$ in M_P , so the connected components of $\mathcal{N}(\partial M_P) \cap \widetilde{M}_P$ inherit coordinates $(\phi, t, \theta) \in S^1 \times (-1, -1/2) \times S^1$.

- \widetilde{M}_Σ is the complement of $\{s \geq -1/2\} \subset \mathcal{N}(\partial M_\Sigma)$ in M_Σ , and we will always use the chosen trivialization of π_Σ to identify this with a subset of $\Sigma \times S^1$, denoting the coordinate on S^1 by θ . The connected components of $\mathcal{N}(\partial M_\Sigma) \cap \widetilde{M}_\Sigma$ thus inherit coordinates $(s, \phi, \theta) \in (-1, -1/2) \times S^1 \times S^1$.
- $\widetilde{M}_\mathcal{I}$ is the union of $\mathcal{N}(\partial M_\Sigma)$ with the components of $\mathcal{N}(\partial M_P)$ that touch M_Σ , so its connected components can each be identified with $(-1, 1) \times S^1 \times S^1$, and we assign coordinates (ρ, ϕ, θ) to these components such that

$$\widetilde{M}_\mathcal{I} \cap M_\Sigma = \{\rho \leq 0\}, \quad \widetilde{M}_\mathcal{I} \cap M_P = \{\rho \geq 0\}.$$

The smooth overlap assumption means we can also assume these coordinates are related to the previously chosen coordinates on these subsets by $\rho = s$ and $\rho = -t$ respectively, with θ and ϕ matching the existing coordinates on $\mathcal{N}(\partial M_\Sigma)$ and $\mathcal{N}(\partial M_P)$ in the obvious way. The region $\widetilde{M}_\mathcal{I}$ will be called the **interface** between the spine and the paper.

- \widetilde{M}_∂ is the union of the components of $\mathcal{N}(\partial M_P)$ that touch ∂M . Its connected components therefore carry collar coordinates $(\phi, t, \theta) \in S^1 \times (-1, 0] \times S^1$, but for consistency with $\widetilde{M}_\mathcal{I}$, we will prefer to use an alternative coordinate system

$$(\rho, \phi, \theta) \in [0, 1) \times S^1 \times S^1 \subset \widetilde{M}_\partial$$

defined by $\rho := -t$.

Note that the above definitions imply

$$\mathcal{N}(\partial M_P) \subset \widetilde{M}_\mathcal{I} \cup \widetilde{M}_\partial,$$

hence we can and sometimes will use the ρ -coordinate as an alternative to the t -coordinate on $\mathcal{N}(\partial M_P)$; they are related by $\rho = -t$.

2.3. Spinal open books support contact structures. We will say that a smooth 1-form α on M is a **fiberwise Giroux form** if the following conditions hold:

- $d\alpha$ is positive on the interior of every page;
- α is positive on the fibers of $\pi_\Sigma : M_\Sigma \rightarrow \Sigma$, and the tangent spaces to these fibers are contained in $\ker d\alpha$;
- At ∂M , α is positive on all boundaries of pages, the tangent spaces to these boundaries are also contained in $\ker d\alpha$, and α vanishes on the foliation \mathcal{F} chosen at the beginning of §2.2.

A fiberwise Giroux form is a Giroux form if and only if it is contact, but since we have not required the latter in the above definition, the space of fiberwise Giroux forms is *convex*. We will show in the following that it is relatively easy to construct fiberwise Giroux forms, and the main idea in the proof of Theorem 1.7 is—following the ideas of Thurston outlined in §2.1—to turn these into Giroux forms by adding large multiples of certain 1-forms pulled back from the bases of the fibrations.

Observe that since every component of Σ has nonempty boundary, we can choose a 1-form σ on Σ satisfying

$$d\sigma > 0 \text{ on } \Sigma, \quad \sigma = e^s d\phi \text{ on } \mathcal{N}(\partial\Sigma).$$

Similarly:

Lemma 2.7. *On M_P there exists a 1-form η such that $d\eta$ is positive on each fiber of $\pi_P : M_P \rightarrow S^1$ and $\eta = e^t d\theta$ in $\mathcal{N}(\partial M_P)$.*

Proof. One only has to observe that since every connected component of $P := \pi_P^{-1}(*)$ has nonempty boundary by assumption, the space of Liouville forms on P which match $e^t d\theta$ in the collars is nonempty and convex. The desired 1-form η can thus be constructed by choosing such a Liouville form η_0 and defining η on each fiber of the mapping torus with monodromy μ as a suitable interpolation between η_0 and $\mu^*\eta_0$; cf. [Etn06, Theorem 3.13]. \square

In order to construct a fiberwise Giroux form, we next choose a smooth function

$$F : M_P \rightarrow (0, 1]$$

which is identically equal to 1 outside of $\mathcal{N}(\partial M_P)$ and takes the form $e^\rho f(\rho)$ in (ρ, ϕ, θ) -coordinates on $\mathcal{N}(\partial M_P) \subset \widetilde{M}_{\mathcal{I}} \cup \widetilde{M}_\partial$, where $f : (-1, 1) \rightarrow (0, 1]$ is a smooth function satisfying the conditions

- $f(\rho) = 1$ for $\rho \leq 0$;
- $f'(\rho) < 0$ for $\rho > 0$;
- $f(\rho) = e^{-\rho}$ for ρ near 1.

In particular, this implies that F admits a smooth extension over $\mathcal{N}(\partial M_\Sigma)$ of the form $F(s, \phi, \theta) = e^s$. Now using the fiberwise Liouville form η provided by Lemma 2.7, we can define a fiberwise Giroux form on M by

$$\alpha = \begin{cases} d\theta & \text{on } M_\Sigma, \\ F\eta & \text{on } M_P. \end{cases}$$

It takes the form $\alpha = f(\rho) d\theta$ on $\widetilde{M}_{\mathcal{I}} \cup \widetilde{M}_\partial$.

We show next how to turn fiberwise Giroux forms into Giroux forms. For any constant $\delta \in (0, 1/2)$, choose a pair of smooth functions $g_\Sigma^\delta, g_\partial^\delta : [0, 1) \rightarrow [0, 2]$ such that

- $g_\Sigma^\delta(\rho) = e^\rho$ for ρ near 0;
- $g_\partial^\delta(0) = 0$ and $(g_\partial^\delta)'(0) > 0$;
- $(g_\Sigma^\delta)'(\rho)$ and $(g_\partial^\delta)'(\rho)$ are both nonnegative for all ρ ;
- $g_\Sigma^\delta(\rho) = g_\partial^\delta(\rho) = 2$ for all $\rho \geq \delta$.

Using this, we define a smooth function $G_\delta : M_P \rightarrow [0, 2]$ by

$$G_\delta = \begin{cases} 2 & \text{on } \widetilde{M}_P, \\ g_\Sigma^\delta(\rho) & \text{on } \mathcal{N}(\partial M_P) \cap \widetilde{M}_{\mathcal{I}}, \\ g_\partial^\delta(\rho) & \text{on } \widetilde{M}_\partial. \end{cases}$$

Then, defining the Liouville form σ as a 1-form on M_Σ by identifying it with its pullback $\pi_\Sigma^*\sigma$, we define for any $\delta \in (0, 1/2)$ another smooth 1-form on M by

$$\beta_\delta = \begin{cases} \sigma & \text{on } M_\Sigma, \\ G_\delta d\phi & \text{on } M_P. \end{cases}$$

Lemma 2.8. *For any fiberwise Giroux form α , there exist constants $\delta_0 \in (0, 1/2)$ and $K_0 \geq 0$ such that for all constants $\delta \in (0, \delta_0]$ and $K \geq K_0$,*

$$\alpha_{K,\delta} := \alpha + K\beta_\delta$$

is a Giroux form. Moreover, whenever α itself is a Giroux form, one can take $K_0 = 0$.

Proof. Observe that $\alpha_{K,\delta}$ is automatically a *fiberwise* Giroux form for all $K \geq 0$, $\delta \in (0, 1/2)$, so we only need to show that $\alpha_{K,\delta}$ is contact for the right choices of these constants. Since $\beta_\delta \wedge d\beta_\delta \equiv 0$, we have

$$\alpha_{K,\delta} \wedge d\alpha_{K,\delta} = K(\alpha \wedge d\beta_\delta + \beta_\delta \wedge d\alpha) + \alpha \wedge d\alpha,$$

thus it suffices to show that whenever $\delta > 0$ is sufficiently small,

$$(2.1) \quad \alpha \wedge d\beta_\delta + \beta_\delta \wedge d\alpha > 0.$$

The conditions on fiberwise Giroux forms imply that $\alpha(\partial_\theta) > 0$ at ∂M_P , so this is also true on collars of the form $\{\rho \leq \delta_0\} \subset \mathcal{N}(\partial M_P)$ for sufficiently small $\delta_0 > 0$. Assuming $0 < \delta \leq \delta_0$, we shall now show that (2.1) holds everywhere on M .

On M_Σ , $\beta_\delta \wedge d\alpha = \sigma \wedge d\alpha = 0$ since $\sigma(\partial_\theta) = d\alpha(\partial_\theta, \cdot) = 0$, but $\alpha \wedge d\beta_\delta > 0$ since $\alpha(\partial_\theta) > 0$ and $d\beta_\delta = d\sigma$ is positive on Σ .

On M_P outside of the collars $\{\rho \leq \delta\}$, we have $\beta_\delta = 2d\phi$ and thus $d\beta_\delta = 0$, while $\beta_\delta \wedge d\alpha = 2d\phi \wedge d\alpha > 0$ due to the assumption that $d\alpha$ is positive on the fibers of π_P .

On the collars $\{\rho \leq \delta\}$, we have $\beta_\delta = G_\delta d\phi$, with $G_\delta > 0$ on the interior of M_P , hence $\beta_\delta \wedge d\alpha = G_\delta d\phi \wedge d\alpha > 0$ again except at ∂M_P . It thus remains only to show that $\alpha \wedge d\beta_\delta \geq 0$, with strict positivity at ∂M_P . This follows from the fact that $\alpha(\partial_\theta) > 0$ on this region, since $\alpha \wedge d\beta_\delta = g'(\rho)\alpha \wedge d\rho \wedge d\phi$, where $g(\rho)$ denotes either $g_\Sigma^\delta(\rho)$ or $g_\partial^\delta(\rho)$, both of which we assumed to have nonnegative first derivatives which are strictly positive at $\rho = 0$. \square

Proof of Theorem 1.7. In light of the construction of a fiberwise Giroux form explained above, the existence of a Giroux form follows immediately from Lemma 2.8.

We claim now that for any $n \in \mathbb{N}$, a continuous family of Giroux forms

$$\{\alpha_\tau\}_{\tau \in S^{n-1}}$$

can always be extended to a family of Giroux forms parametrized by the disk \mathbb{D}^n . As an initial step, note that the characteristic foliations induced by α_τ at ∂M may not be precisely the foliation \mathcal{F} we fixed above, but they are guaranteed to be isotopic to it and also transverse to the coordinate vector field ∂_θ (which is parallel to Reeb orbits at the boundary). We can thus alter α_τ by a fiber preserving isotopy supported near ∂M , producing a homotopy through S^{n-1} -families of Giroux forms, to a family whose characteristic foliations at ∂M are all generated by ∂_ϕ . Let us therefore assume without loss of generality that the given family α_τ has this property, so all the α_τ are also fiberwise Giroux forms by our definition.

Since the space of fiberwise Giroux forms is convex, $\{\alpha_\tau\}_{\tau \in S^{n-1}}$ can now be extended via linear interpolation to a family $\{\tilde{\alpha}_\tau\}_{\tau \in \mathbb{D}^n}$ of fiberwise Giroux forms. These forms are also contact for all τ in some collar neighborhood of $\partial \mathbb{D}^n$, since the contact condition is open. Choose a continuous ‘‘bump’’ function

$$\psi : \mathbb{D}^n \rightarrow [0, 1]$$

that equals 0 at $\partial \mathbb{D}^n$ and 1 outside this collar. Next, observe that since \mathbb{D}^n is compact, one can find constants $K \geq 0$ sufficiently large and $\delta > 0$ sufficiently small so that Lemma 2.8 holds with the same constants for all $\tilde{\alpha}_\tau$, $\tau \in \mathbb{D}^n$. Then

$$\alpha_\tau := \tilde{\alpha}_\tau + K\psi(\tau)\beta_\delta$$

defines the desired family of Giroux forms. This shows that the space of Giroux forms has vanishing homotopy groups of all orders, so by Whitehead’s theorem, it is contractible. \square

We can now fill in a loose end from §1.3 and complete the proof of Proposition 1.32.

Lemma 2.9. *Every contact manifold with planar 1-torsion is overtwisted.*

Proof. Suppose (M, ξ) contains a planar 0-torsion domain M_0 , so (M_0, ξ) is supported by a spinal open book π whose interior contains a page D that is a disk. Let $M_\Sigma^1 \subset M_0$ denote the spinal region adjacent to D . Since π is not symmetric, there is a paper component $M_P^1 \subset M_0$ adjacent to M_Σ^1 with a page $P_1 \subset M_P^1$ that is not a disk. Pick an embedded curve L in the interior of P_1 that is smoothly isotopic to a boundary component adjacent to M_Σ^1 , so L is also smoothly isotopic to the boundary of D , and both page framings agree and are equal to 0. We will show that one can realize L as a Legendrian knot with Thurston-Bennequin number $\text{tb}(L) = 0$, violating the Bennequin-Eliashberg bound if (M, ξ) is tight.

We consider two cases. First, assume that P_1 has another boundary component. Then in the construction of the Giroux form, the 1-form η of Lemma 2.7 can be chosen to vanish identically along L . Choosing the support of the monodromy away from L , it remains Legendrian after converting η into the compatible contact form α , and the contact framing is 0 relative to P_1 , and hence also relative to the disk D .

Alternatively, suppose P_1 has a single boundary component (to which L is isotopic), and since π is not symmetric, P_1 has genus $g > 0$. Since P_1 is a convex surface (but with transverse boundary), we can flow it along a transverse contact vector field to create a neighborhood of the form $P_1 \times [0, 1] \subset M_P^1$ and round the corners to produce a convex handlebody whose dividing set is isotopic to $\partial P_1 \times \{1/2\}$. On this convex surface, L is isolating (in the sense of Honda [Hon00]), but since $g > 0$, we can fold along any other (disjoint, homotopically nontrivial, embedded) curve in $P_1 \times \{1\}$, increasing the dividing set and making L non-isolating. We can now Legendrian realize L , and since L is disjoint from the dividing set, we can ensure this has contact framing 0 relative to $P_1 \times \{1\}$. This is an absolute 0-framing since the framings from $P_1 \times \{1\}$, P_1 and the disk D all agree. \square

2.4. Lefschetz fibrations and symplectic structures. In this section we prove the main part of Theorem 1.24 regarding the various spaces of symplectic structures supported by a bordered Lefschetz fibration. The overall strategy is similar to that of the previous section, and can be summarized as follows:

- (1) Define spaces of “fiberwise” symplectic structures which are manifestly contractible, and are nonempty under suitable assumptions.
- (2) Use the Thurston trick to turn fiberwise structures into supported symplectic structures by adding large multiples of data pulled back from the base.

A version of Theorem 1.24 for almost Stein structures appeared already in our appendix to [BV15], and we will repeat some of those arguments here but will generalize them substantially in §3 below, with an eye toward classifying fillings up to Stein homotopy.

For this subsection and the next, fix a bordered Lefschetz fibration $\Pi : E \rightarrow \Sigma$. Recall that a symplectic structure ω on E was defined to be *supported* by Π if it is positive on fibers and also tames some almost complex structure J defined near the critical points E^{crit} for which the fibers are J -holomorphic. It will be useful to note that this last condition doesn’t depend on the choice of J :

Proposition 2.10. *Suppose J_1 and J_2 are two almost complex structures defined near E^{crit} which each restrict to positively oriented complex structures on the smooth part of every fiber. Then $J_1|_{TE^{\text{crit}}} = J_2|_{TE^{\text{crit}}}$.*

Proof. By [Gom04, Lemma 4.4(a)], it will suffice to observe that J_1 and J_2 determine the same oriented complex 1-dimensional subspaces in $TE|_{E^{\text{crit}}}$. Indeed, choosing local complex

coordinates (z_1, z_2) near a point $p \in E^{\text{crit}}$ and a corresponding complex coordinate near $\Pi(p)$ such that $\Pi(z_1, z_2) = z_1^2 + z_2^2$, we see that in these coordinates every complex 1-dimensional subspace of \mathbb{C}^2 occurs as a tangent space to a fiber in any neighborhood of p . Since such tangent spaces are both J_1 - and J_2 -complex by assumption, the claim follows by continuity. \square

In the following, fix an integrable complex structure J^{crit} near E^{crit} for which Π is holomorphic near E^{crit} . Proposition 2.10 implies that none of our definitions or results will depend on this choice. We shall now define various spaces of smooth objects on E , each assumed to carry the natural C^∞ -topology. Denote the vertical subbundles in E and $\partial_h E$ by

$$VE = \ker T\Pi \subset TE \quad \text{and} \quad V(\partial_h E) = VE \cap T(\partial_h E).$$

Definition 2.11. Let $\Lambda^{\text{fib}}(\partial_h \Pi)$ denote the space of germs of 1-forms λ defined on a neighborhood of $\partial_h E$ in E such that $\lambda|_{V(\partial_h E)} > 0$ and $V(\partial_h E) \subset \ker(d\lambda|_{T(\partial_h E)})$. Similarly, $\Lambda^{\text{fib}}(\partial \Pi)$ will denote the space of germs of 1-forms λ defined on a neighborhood of ∂E in E which satisfy the above conditions at $\partial_h E$ and also satisfy $d\lambda|_{VE} > 0$ at $\partial_v E$. We call any $\lambda \in \Lambda^{\text{fib}}(\partial_h \Pi)$ or $\Lambda^{\text{fib}}(\partial \Pi)$ a **fiberwise Giroux form near $\partial_h E$ or ∂E respectively**.

Observe that $\Lambda^{\text{fib}}(\partial_h \Pi)$ and $\Lambda^{\text{fib}}(\partial \Pi)$ are both convex spaces.

Definition 2.12. The spaces of **Giroux forms near $\partial_h E$ or ∂E respectively** (cf. Remark 1.13) are defined as

$$\begin{aligned} \Lambda(\partial_h \Pi) &:= \left\{ \lambda \in \Lambda^{\text{fib}}(\partial_h \Pi) \mid \lambda|_{T(\partial_h E)} \text{ is contact} \right\}, \\ \Lambda(\partial \Pi) &:= \left\{ \lambda \in \Lambda^{\text{fib}}(\partial \Pi) \mid \lambda|_{T(\partial_h E)} \text{ and } \lambda|_{T(\partial_v E)} \text{ are both contact} \right\}. \end{aligned}$$

The following variation on Theorem 1.7 follows from a simpler version of the same argument, implementing the Thurston trick via Propositions 2.1 and 2.2. It implies in particular that both $\Lambda(\partial_h \Pi)$ and $\Lambda(\partial \Pi)$ are nonempty and contractible.

Proposition 2.13. *The spaces $\Lambda^{\text{fib}}(\partial_h \Pi)$ and $\Lambda^{\text{fib}}(\partial \Pi)$ are each nonempty. Moreover, fixing a Liouville form σ on Σ , for any $\lambda \in \Lambda^{\text{fib}}(\partial_h \Pi)$ or $\Lambda^{\text{fib}}(\partial \Pi)$, there exists a constant $K_0 \geq 0$ depending continuously on λ such that for every constant $K \geq K_0$, $\lambda + K\Pi^*\sigma$ belongs to $\Lambda(\partial_h \Pi)$ or $\Lambda(\partial \Pi)$ respectively, and we can take $K_0 = 0$ if λ is already in $\Lambda(\partial_h \Pi)$ or $\Lambda(\partial \Pi)$. \square*

Definition 2.14. The space of **weakly convex fiberwise symplectic** structures $\Omega_{\text{weak}}^{\text{fib}}(\Pi)$ consists of all smooth closed 2-forms ω on E such that

- (1) ω is positive on all fibers in $E \setminus E^{\text{crit}}$;
- (2) At E^{crit} , ω is nondegenerate and tames J^{crit} ;
- (3) Near $\partial_h E$, $\omega = d\lambda$ for some $\lambda \in \Lambda^{\text{fib}}(\partial_h \Pi)$.

The space of supported weakly convex symplectic structures is then

$$\Omega_{\text{weak}}(\Pi) := \left\{ \omega \in \Omega_{\text{weak}}^{\text{fib}}(\Pi) \mid \omega^2 > 0 \text{ and } \omega = d\lambda \text{ near } \partial_h E \text{ for some } \lambda \in \Lambda(\partial_h \Pi) \right\}.$$

The space of **strongly convex fiberwise symplectic** structures will be

$$\Omega_{\text{strong}}^{\text{fib}}(\Pi) := \left\{ \omega \in \Omega_{\text{weak}}^{\text{fib}}(\Pi) \mid \omega = d\lambda \text{ near } \partial E \text{ for some } \lambda \in \Lambda^{\text{fib}}(\partial \Pi) \right\},$$

so that the space of supported strongly convex symplectic structures is

$$\Omega_{\text{strong}}(\Pi) := \left\{ \omega \in \Omega_{\text{strong}}^{\text{fib}}(\Pi) \mid \omega^2 > 0 \text{ and } \omega = d\lambda \text{ near } \partial E \right. \\ \left. \text{for some } \lambda \in \Lambda(\partial\Pi) \right\}.$$

We similarly define the space of **fiberwise Liouville structures** $\Omega_{\text{exact}}^{\text{fib}}(\Pi)$ to consist of all $\omega \in \Omega_{\text{strong}}^{\text{fib}}(\Pi)$ for which the primitive $\lambda \in \Lambda^{\text{fib}}(\partial\Pi)$ extends to a global primitive of ω (i.e. a **fiberwise Liouville form**) on E . The space of supported Liouville structures is then

$$\Omega_{\text{exact}}(\Pi) := \left\{ \omega \in \Omega_{\text{exact}}^{\text{fib}}(\Pi) \mid \omega^2 > 0 \text{ and } \omega = d\lambda \text{ on } E \right. \\ \left. \text{for some } \lambda \text{ with } \lambda|_{\partial E} \in \Lambda(\partial\Pi) \right\}.$$

Observe that there are natural inclusions

$$\Omega_{\text{exact}}^{\text{fib}}(\Pi) \hookrightarrow \Omega_{\text{strong}}^{\text{fib}}(\Pi) \hookrightarrow \Omega_{\text{weak}}^{\text{fib}}(\Pi),$$

and all three spaces are convex.

To handle the Stein case, we shall consider a special space of almost complex structures. Given any almost complex structure J on E , denote the maximal J -complex subbundle in $T(\partial_h E)$ by

$$\xi_J := T(\partial_h E) \cap JT(\partial_h E) \subset T(\partial_h E).$$

Definition 2.15. Let $\mathcal{J}(\Pi)$ denote the space of pairs (J, ∂_θ) where J is an almost complex structure on E compatible with its orientation, ∂_θ is a nowhere zero vertical vector field on $\partial_h E$, oriented in the positive direction of the fibers, and the following properties are satisfied:

- (1) There exists a complex structure j on Σ for which $\Pi : (E, J) \rightarrow (\Sigma, j)$ is pseudoholomorphic;
- (2) The flow of ∂_θ is 1-periodic and preserves ξ_J .

Note that any $(J, \partial_\theta) \in \mathcal{J}(\Pi)$ uniquely determines j on Σ . The choice of vector field ∂_θ is equivalent to a choice of principal S^1 -bundle structure on $\partial_h E$, so it defines a fiber-preserving S^1 -action that preserves both ξ_J and (due to the first condition) $J|_{\xi_J}$. We do not require J to match J^{crit} near E^{crit} , though they automatically match at E^{crit} due to Proposition 2.10. Using the fact that the space of positively oriented complex structures on any oriented real vector bundle of rank 2 is nonempty and contractible, it follows that the same is true for $\mathcal{J}(\Pi)$.

Remark 2.16. In contact geometric terms, defining a principal S^1 -bundle structure on $\partial_h E$ is equivalent to giving it the structure of a *strict* contact fiber bundle (cf. Remark 2.3), i.e. each fiber is identified with the contact manifold (S^1, dt) so that the vector field ∂_θ generating the S^1 -action satisfies $dt(\partial_\theta) \equiv 1$. Any fiberwise Giroux form $\lambda \in \Lambda^{\text{fib}}(\partial_h \Pi)$ near $\partial_h E$ canonically determines a strict contact fiber bundle structure, with a positive constant multiple of λ as the contact form on each fiber; here the condition $d\lambda(\partial_\theta, \cdot)|_{T(\partial_h E)} \equiv 0$ ensures that all fibers are strictly contactomorphic since λ has the same integral on all of them, by Stokes' theorem.

Definition 2.17. Given any $(J, \partial_\theta) \in \mathcal{J}(\Pi)$, we will say that a smooth function $f : E \rightarrow \mathbb{R}$ is **fiberwise J -convex** if, writing $\lambda_J := -df \circ J$, the following conditions are satisfied:

- (1) f is constant on each boundary component of each fiber $E_z \subset E$;
- (2) $d\lambda_J \in \Omega_{\text{exact}}^{\text{fib}}(\Pi)$;
- (3) $\lambda_J|_{\partial E} \in \Lambda^{\text{fib}}(\partial\Pi)$;
- (4) $\lambda_J(\partial_\theta)$ is constant.

The space of fiberwise J -convex functions for a fixed $(J, \partial_\theta) \in \mathcal{J}(\Pi)$ will be denoted by $PSH_{(J, \partial_\theta)}^{fib}(\Pi)$.

Observe that $PSH_{(J, \partial_\theta)}^{fib}(\Pi)$ is convex for each $(J, \partial_\theta) \in \mathcal{J}(\Pi)$, and there is a natural map

$$PSH_{(J, \partial_\theta)}^{fib}(\Pi) \rightarrow \Omega_{\text{exact}}^{fib}(\Pi) : f \mapsto -d(df \circ J).$$

Definition 2.18. Let $PSH_{(J, \partial_\theta)}(\Pi) \subset PSH_{(J, \partial_\theta)}^{fib}(\Pi)$ denote the subspace for which $d\lambda_J$ is also a symplectic form taming J and $\lambda_J|_{\partial E} \in \Lambda(\partial\Pi)$.

The space of supported almost Stein structures is now precisely

$$\mathcal{J}_{\text{Stein}}(\Pi) = \{(J, f) \mid (J, \partial_\theta) \in \mathcal{J}(\Pi) \text{ for some } \partial_\theta, \text{ and } f \in PSH_{(J, \partial_\theta)}(\Pi)\}.$$

Note that for any $(J, f) \in \mathcal{J}_{\text{Stein}}(\Pi)$, the vector field $\hat{\partial}_\theta$ is canonically determined via Remark 2.16, hence there is a well-defined projection

$$(2.2) \quad \mathcal{J}_{\text{Stein}}(\Pi) \rightarrow \mathcal{J}(\Pi) : (J, f) \mapsto (J, \partial_\theta),$$

whose fiber over any $(J, \partial_\theta) \in \mathcal{J}(\Pi)$ is $PSH_{(J, \partial_\theta)}(\Pi)$. Since $\mathcal{J}(\Pi)$ is homotopy equivalent to a point, the almost Stein part of Theorem 1.24 will then be a consequence of the following statement, to be proved at the very end of this subsection:

Proposition 2.19. *If $\Pi : E \rightarrow \Sigma$ is allowable, then the projection (2.2) is a Serre fibration with contractible fibers; in particular, it is a homotopy equivalence.*

Remark 2.20. The contractibility of $PSH_{(J, \partial_\theta)}(\Pi)$ for each $(J, \partial_\theta) \in \mathcal{J}(\Pi)$ is not as obvious as it may at first appear, e.g. since functions in $PSH_{(J, \partial_\theta)}(\Pi)$ are not constant at the boundary, $PSH_{(J, \partial_\theta)}(\Pi)$ is not generally convex (cf. the discussion of almost Stein structures preceding Definition 1.23). The proof that $PSH_{(J, \partial_\theta)}(\Pi)$ is contractible will instead require the Thurston trick.

We will frequently need to use the following standard lemma in constructions of J -convex functions. Recall that a hypersurface V in an almost complex manifold (W, J) is called **J -convex** whenever the maximal J -complex subbundle in TV is a contact structure whose canonical conformal symplectic structure tames J .

Lemma 2.21 (see e.g. [CE12, Lemma 2.7] or [LW11, Lemma 4.1]). *Suppose (W, J) is a smooth almost complex manifold and $f : W \rightarrow \mathbb{R}$ is a smooth function such that f is J -convex near all its critical points and all level sets of f are J -convex hypersurfaces wherever they are regular. Then if $h : \mathbb{R} \rightarrow \mathbb{R}$ is any smooth function with $h' > 0$ and h'' everywhere sufficiently large, $h \circ f$ is a J -convex function. \square*

Remark 2.22. It will sometimes be useful to note that the J -convexity hypothesis on hypersurfaces is vacuous when $\dim_{\mathbb{R}} W = 2$.

In order to construct fiberwise symplectic structures in the nonexact case, we will need first to be able to pick a cohomology class that evaluates positively on every irreducible component of every fiber. For this we will make use of the following linear algebraic lemma due to Gompf.

Lemma 2.23 ([Gom05, Lemma 3.3]). *For a real n -by- n symmetric matrix $A = (a_{ij})$, let G_A denote the graph with n vertices v_1, \dots, v_n , and an edge between any two distinct vertices v_i, v_j whenever $a_{ij} \neq 0$. Suppose that (a) G_A is connected, (b) $a_{ij} \geq 0$ whenever $i \neq j$, and (c) there are positive real numbers m_1, \dots, m_n such that $\sum_{i=1}^m m_i a_{ij} \leq 0$ for all j . Fix a choice*

of such numbers m_i . Then the hypothesis (d), that the inequality in (c) is strict for some j , implies $\text{rank } A = n$. If (d) is not satisfied, then $\text{rank } A = n - 1$. \square

Lemma 2.24. *Given any $\lambda \in \Lambda^{\text{fib}}(\partial_h \Pi)$ or $\Lambda^{\text{fib}}(\partial \Pi)$, there exists a closed 2-form η on E such that $\eta = d\lambda$ near $\partial_h E$ or ∂E respectively and $\int_C \eta > 0$ for every irreducible component C of every fiber.*

Proof. Extending λ arbitrarily to a smooth 1-form on E , Stokes' theorem implies $\int_C d\lambda \geq 0$ for all irreducible components C of fibers, with strict inequality if and only if $\partial C \neq \emptyset$. Our main task will be to find a closed 2-form ω supported in the interior such that $\int_C \omega > 0$ for every component C with $\partial C = \emptyset$, as we can then set $\eta := d\lambda + \epsilon\omega$ for sufficiently small $\epsilon > 0$.

We construct ω as follows. The collection of all closed irreducible components of singular fibers defines a graph Γ , with vertices corresponding to closed irreducible components and edges corresponding to critical points at which two such components intersect each other. Pick any connected component of Γ and denote the corresponding closed irreducible components of fibers by C_1, \dots, C_n . Pick closed 2-forms $\omega_1, \dots, \omega_n$ such that for $i = 1, \dots, n$, ω_i represents the Poincaré dual of $[C_i]$ and is supported in a neighborhood of C_i disjoint from ∂E . Let \hat{n}_i denote the number of critical points at which C_i intersects other irreducible components (i.e. not counting intersections of C_i with itself), and let $n_i \leq \hat{n}_i$ denote the number of these at which C_i intersects other *closed* components (this is the number of edges touching the corresponding vertex in Γ). For $i, j \in 1, \dots, n$, let n_{ij} denote the number of critical points at which C_i and C_j intersect, i.e. the number of edges of Γ connecting the two corresponding vertices. The algebraic intersection numbers $[C_i] \cdot [C_j] \in \mathbb{Z}$ then satisfy

$$\begin{aligned} [C_i] \cdot [C_j] &= n_{ij} \text{ for } i \neq j, \\ [C_i] \cdot [C_i] &= -\hat{n}_i, \end{aligned}$$

thus for $i = 1, \dots, n$,

$$(2.3) \quad \sum_{j=1}^n [C_i] \cdot [C_j] = \sum_{j \neq i} n_{ij} - \hat{n}_i = n_i - \hat{n}_i \leq 0.$$

Since no fiber consists exclusively of closed components, the inequality $n_i - \hat{n}_i \leq 0$ must be strict for some $i = 1, \dots, n$.

Define now an n -by- n symmetric matrix $A = (a_{ij})$ with entries $a_{ij} = [C_i] \cdot [C_j]$. By (2.3), A satisfies the conditions of Lemma 2.23 with $m_1 = \dots = m_n = 1$, including hypothesis (d), hence $\text{rank } A = n$. It follows that one can find coefficients $b_1, \dots, b_n \in \mathbb{R}$ such that

$$\int_{C_i} \sum_{j=1}^n b_j \omega_j = \sum_{j=1}^n a_{ij} b_j > 0$$

for all $i = 1, \dots, n$. The desired 2-form ω can thus be defined as a sum of 2-forms of this type for each connected component of the graph Γ . \square

The next proposition is the main existence result for fiberwise symplectic structures.

Proposition 2.25. *Given any $\lambda \in \Lambda^{\text{fib}}(\partial_h \Pi)$ or $\Lambda^{\text{fib}}(\partial \Pi)$, there exists $\omega \in \Omega_{\text{strong}}^{\text{fib}}(\Pi)$ such that $\omega = d\lambda$ near $\partial_h E$ or ∂E respectively. In particular, the spaces $\Omega_{\text{weak}}^{\text{fib}}(\Pi)$ and $\Omega_{\text{strong}}^{\text{fib}}(\Pi)$ are always nonempty. Moreover, for any $(J, \partial_\theta) \in \mathcal{J}(\Pi)$, $\text{PSH}_{(J, \partial_\theta)}^{\text{fib}}(\Pi)$ (and hence also $\Omega_{\text{exact}}^{\text{fib}}(\Pi)$) is nonempty if and only if Π is allowable.*

Proof. If Π is not allowable, then there is a closed component in some singular fiber, thus Stokes' theorem implies there can be no exact 2-form that is positive on every fiber. Consequently, $\Omega_{\text{exact}}^{\text{fib}}(\Pi)$ (and therefore also $PSH_{(J, \partial_\theta)}^{\text{fib}}(\Pi)$) must be empty.

In the following, we shall handle the construction of $\omega \in \Omega_{\text{strong}}^{\text{fib}}(\Pi)$ and $f \in PSH_{(J, \partial_\theta)}^{\text{fib}}(\Pi)$ in parallel at each step. The construction of $f \in PSH_{(J, \partial_\theta)}^{\text{fib}}(\Pi)$ depends on an arbitrary choice of $(J, \partial_\theta) \in \mathcal{J}(\Pi)$, which we shall assume fixed throughout. Note that while $\omega \in \Omega_{\text{strong}}^{\text{fib}}(\Pi)$ is required to match $d\lambda$ for a prescribed primitive λ near the boundary, the statement for the almost Stein case does not require this.

Given $\lambda \in \Lambda^{\text{fib}}(\partial_h \Pi)$ or $\Lambda^{\text{fib}}(\partial \Pi)$, let η denote the closed 2-form guaranteed by Lemma 2.24. Observe that by the fiberwise Giroux form condition, the integrals of λ over boundary components of fibers E_z are locally constant functions of z . We proceed in three steps:

Step 1: Neighborhoods of regular fibers. For each $z \in \Sigma \setminus \Sigma^{\text{crit}}$, there exists an open neighborhood $z \in \mathcal{U}_z \subset \Sigma \setminus \Sigma^{\text{crit}}$ and a 1-form λ_z on $E|_{\mathcal{U}_z}$ which restricts to λ near $\partial_h E$ such that $\omega_z := d\lambda_z$ is an area form on every fiber in $E|_{\mathcal{U}_z}$. If $\lambda \in \Lambda^{\text{fib}}(\partial \Pi)$ then $d\lambda$ is already positive on the fibers near $\partial_v E$, thus we can also arrange $\lambda_z = \lambda$ near $\partial_v E$.

For the almost Stein case, observe first that the vector field $-J\partial_\theta$ along $\partial_h E$ is necessarily vertical and points transversely outward. Choose a smooth function $f_z : E_z \rightarrow \mathbb{R}$ such that $-d(df_z \circ J) > 0$ on E_z , while at ∂E_z , $f_z \equiv c_z$ and $df_z(-J\partial_\theta) \equiv \nu_z$ for some constants $c_z, \nu_z > 0$. This can be achieved by starting with any smooth function that satisfies these conditions at the boundary and has only Morse critical points of index 0 and 1, then modifying it by local diffeomorphisms to make it J -convex near the critical points, and postcomposing it with a function with very large second derivative; cf. Lemma 2.21. We can then find a neighborhood $\mathcal{U}_z \subset \Sigma \setminus \Sigma^{\text{crit}}$ of z such that f_z admits an extension to a function

$$f_z : E|_{\mathcal{U}_z} \rightarrow \mathbb{R}$$

satisfying these same properties on every fiber in $E|_{\mathcal{U}_z}$. Note that the constants c_z and ν_z can always be made larger without changing the neighborhood \mathcal{U}_z . The 1-form $\lambda_z := -df_z \circ J$ on $E|_{\mathcal{U}_z}$ now satisfies $d\lambda_z > 0$ on every fiber, and its restriction to the horizontal boundary

$$\alpha_z^h := \lambda_z|_{T(\partial_h E)}$$

satisfies $\alpha_z^h(\partial_\theta) = \nu_z$, $\alpha_z^h|_{\xi_J} = 0$. The invariance of ξ_J under the flow of ∂_θ then implies $\mathcal{L}_{\partial_\theta} \alpha_z^h = d\alpha_z^h(\partial_\theta, \cdot) = 0$.

Step 2: Neighborhoods of singular fibers. For each $z \in \Sigma^{\text{crit}}$, let E_z^{crit} denote the finite set of critical points in E_z . For each $p \in E_z^{\text{crit}}$, choose J^{crit} -holomorphic Morse coordinates (z_1, z_2) on a neighborhood $\mathcal{U}_p \subset E$ of p , and let ω^{crit} denote the symplectic form on \mathcal{U}_p which looks like the standard symplectic form on \mathbb{C}^2 in these coordinates. Choose an area form ω_z on $E_z \setminus E_z^{\text{crit}}$ satisfying the following conditions:

- ω_z restricts to $d\lambda$ near $\partial_h E$;
- $\omega_z = \omega^{\text{crit}}$ near E_z^{crit} ;
- For each irreducible component $C \subset E_z$, $\int_C \omega_z = \int_C \eta$.

This can be extended to a closed 2-form on $E|_{\mathcal{U}_z}$ for some open neighborhood $z \in \mathcal{U}_z \subset \Sigma$ with $\overline{\mathcal{U}_z} \subset \overset{\circ}{\Sigma}$, such that the extended ω_z also matches $d\lambda$ near $\partial_h E$ and is positive on fibers.

For the almost Stein case, we must assume explicitly at this step that $\Pi : E \rightarrow \Sigma$ is allowable, so in particular, the connected components of $E_z \setminus E_z^{\text{crit}}$ are all compact oriented surfaces with *nonempty boundary* and finitely many punctures. Using the same J^{crit} -holomorphic

coordinates (z_1, z_2) as above near any $p \in E_z^{\text{crit}}$, define a function $f_z : \mathcal{U}_p \rightarrow \mathbb{R}$ by

$$f_z(z_1, z_2) = \frac{1}{2} (|z_1|^2 + |z_2|^2).$$

This function is J^{crit} -convex, and we claim that it is also J -convex on a sufficiently small neighborhood of p . To see this, recall that J and J^{crit} match at p due to Proposition 2.10. Since $df_z(p) = 0$, the 1-forms $-df_z \circ J$ and $-df_z \circ J^{\text{crit}}$ have the same 1-jet at p , so their exterior derivatives match at that point, and the claim follows. By shrinking \mathcal{U}_p if necessary, we can therefore assume $-df_z \circ J$ is the primitive of a positive symplectic form in \mathcal{U}_p that tames J and restricts symplectically to the vertical subspaces. Now since every connected component of $E_z \setminus E_z^{\text{crit}}$ has nonempty boundary, we can extend f_z over E_z so that it is J -convex on E_z and satisfies $f_z \equiv c_z$, $df_z(-J\partial_\theta) \equiv \nu_z$ at ∂E_z . Using the fact that J -convexity is an open condition, we can then extend f_z over $E|_{\mathcal{U}_z}$ for some neighborhood $z \in \mathcal{U}_z \subset \Sigma$ so that it has these same properties on each fiber. The constants c_z and ν_z can again be made larger if desired without changing the neighborhood \mathcal{U}_z .

Step 3: Partition of unity. Since Σ is compact, there is a finite subset $I \subset \Sigma$ such that the open sets $\{\mathcal{U}_z\}_{z \in I}$ cover Σ . Choose a partition of unity $\{\rho_z : \mathcal{U}_z \rightarrow [0, 1]\}_{z \in I}$ subordinate to this cover. For each $z \in I$, the 2-form $\omega_z - \eta$ on $E|_{\mathcal{U}_z}$ is exact by construction, thus we can pick a 1-form θ_z on $E|_{\mathcal{U}_z}$ with

$$\omega_z = \eta + d\theta_z,$$

and since ω_z and η both match $d\lambda$ on a neighborhood of $\partial_h E$ or ∂E respectively, we can choose θ_z such that $\theta_z = 0$ on such a neighborhood. We can then define $\omega \in \Omega_{\text{strong}}^{\text{fib}}(\Pi)$ by

$$\omega = \eta + d \left(\sum_{z \in I} (\rho_z \circ \Pi) \theta_z \right).$$

For the almost Stein case, consider the same partition of unity with the functions $f_z : E|_{\mathcal{U}_z} \rightarrow \mathbb{R}$ constructed in the first two steps, for $z \in I$. By making these functions more convex near $\partial_h E$, we can increase the constants $c_z > 0$ for all $z \in I$ so that they match a single constant $c > 0$, and likewise increase ν_z for $z \in I$ to match some large number $\nu > 0$. The function

$$f := \sum_{z \in I} (\rho_z \circ \Pi) f_z$$

is then constant at $\partial_h E$. Writing $\lambda_J = -df \circ J$, we also have $d\lambda_J > 0$ on all fibers, while $d\lambda_J$ is symplectic and tames J near E^{crit} , and the 1-form $\alpha^h := \lambda_J|_{T(\partial_h E)}$ satisfies

$$\alpha^h(\partial_\theta) \equiv \nu > 0, \quad \text{and} \quad \alpha^h|_{\xi_J} \equiv 0,$$

thus the invariance of ξ_J under the flow of ∂_θ implies

$$d\alpha^h(\partial_\theta, \cdot) \equiv \mathcal{L}_{\partial_\theta} \alpha^h \equiv 0.$$

□

Remark 2.26. It will occasionally (e.g. in Lemma 3.12) be useful to observe that in the almost Stein case, the above proof did not make any use of the assumption that $\Pi : (E, J) \rightarrow (\Sigma, j)$ is pseudoholomorphic. The conditions on (J, ∂_θ) we used were merely that every fiber is J -holomorphic and the S^1 -action defined by ∂_θ on $\partial_h E$ preserves $\xi_J := T(\partial_h E) \cap JT(\partial_h E)$ and $J|_{\xi_J}$.

To move from fiberwise structures to honest symplectic structures, we apply the Thurston trick. Fix a Liouville form σ on Σ . For the almost Stein case, we may also assume

$$\sigma = -d\varphi \circ j,$$

where $\varphi : \Sigma \rightarrow \mathbb{R}$ is a smooth function constant at the boundary and j is the unique complex structure on Σ for which $\Pi : (E, J) \rightarrow (\Sigma, j)$ is pseudoholomorphic.

Proposition 2.27. *Given ω in $\Omega_{\text{weak}}^{\text{fib}}(\Pi)$, $\Omega_{\text{strong}}^{\text{fib}}(\Pi)$ or $\Omega_{\text{exact}}^{\text{fib}}(\Pi)$, there exists a constant $K_0 \geq 0$, depending continuously on ω , such that for every $K \geq K_0$,*

$$\omega_K := \omega + K \Pi^* d\sigma$$

belongs to $\Omega_{\text{weak}}(\Pi)$, $\Omega_{\text{strong}}(\Pi)$ or $\Omega_{\text{exact}}(\Pi)$ respectively.

Similarly, given $(J, \partial_\theta) \in \mathcal{J}(\Pi)$ and $f \in \text{PSH}_{(J, \partial_\theta)}^{\text{fib}}(\Pi)$, there exists $K_0 \geq 0$, depending continuously on J and f , such that for every $K \geq K_0$,

$$f_K := f + K(\varphi \circ \Pi)$$

belongs to $\text{PSH}_{(J, \partial_\theta)}(\Pi)$.

Moreover, if ω is already in $\Omega_{\text{weak}}(\Pi)$, $\Omega_{\text{strong}}(\Pi)$ or $\Omega_{\text{exact}}(\Pi)$, or f is already in $\text{PSH}_{(J, \partial_\theta)}(\Pi)$ respectively, then for both statements it suffices to set $K_0 = 0$.

Proof. Let $\mathcal{U}^{\text{crit}} \subset E$ denote a neighborhood of E^{crit} on which the integrable complex structure J^{crit} is defined and $\Pi|_{\mathcal{U}^{\text{crit}}}$ is holomorphic; more precisely for each $p \in E^{\text{crit}}$, a neighborhood of $\Pi(p)$ in Σ admits a complex structure j_p such that the restriction of Π to the connected component \mathcal{U}_p of $\mathcal{U}^{\text{crit}}$ containing p is a holomorphic map $(\mathcal{U}_p, J^{\text{crit}}) \rightarrow (\Pi(\mathcal{U}_p), j_p)$. Assume to start with that $\omega \in \Omega_{\text{weak}}^{\text{fib}}(\Pi)$. By shrinking $\mathcal{U}^{\text{crit}}$ if necessary, we may assume $\omega|_{\mathcal{U}^{\text{crit}}}$ is symplectic and tames J^{crit} . Now for any nonzero vector $v \in TE|_{\mathcal{U}_p}$ for $p \in E^{\text{crit}}$, we have

$$\omega_K(v, J^{\text{crit}}v) = \omega(v, J^{\text{crit}}v) + K d\sigma(\Pi_*v, j_p \Pi_*v),$$

in which the first term is positive and the second is nonnegative for any $K \geq 0$, hence $\omega_K|_{\mathcal{U}^{\text{crit}}}$ is symplectic and positive on the fibers. In the almost Stein case, we write $\lambda_J := -df \circ J$ and $\sigma := -d\varphi \circ j$ and observe that the holomorphicity of Π implies $-d(\varphi \circ \Pi) \circ J = \Pi^*(-d\varphi \circ j) = \Pi^*\sigma$, hence $\lambda_J^K := -df_K \circ J = \lambda + K \Pi^*\sigma$. We then have

$$d\lambda_J^K(v, Jv) = d\lambda_J(v, Jv) + K d\sigma(\Pi_*v, j \Pi_*v),$$

and for any $v \neq 0$ near E^{crit} this is again positive since $d\lambda_J$ tames J^{crit} and, by Prop. 2.10, the latter matches J at E^{crit} .

Outside a neighborhood of E^{crit} , the rest follows by direct application of the results in §2.1. \square

Applying Whitehead's theorem as in the proof of Theorem 1.7, Propositions 2.25 and 2.27 together imply that the various spaces of supported symplectic structures in Theorem 1.24 are nonempty and contractible as claimed. They also imply that the fibers of the projection $\mathcal{J}_{\text{Stein}}(\Pi) \rightarrow \mathcal{J}(\Pi) : (J, f) \mapsto (J, \partial_\theta)$ are nonempty and contractible. To see that this projection is also a Serre fibration, it suffices to observe that due to the continuous dependence on J and f , the construction in Proposition 2.27 of the J -convex function f_K can be done parametrically. This completes the proof of Proposition 2.19.

2.5. Smoothing corners. To finish the proof of Theorem 1.24, we must show that the corners of ∂E can be smoothed in a way that yields a symplectic filling canonically up to deformation. For strong fillings this is mostly obvious because we have a Liouville vector field transverse to both smooth faces of ∂E , but the case of weak fillings requires a bit more thought since there is no Liouville vector field. We will consider a specific class of smoothings defined as follows.

Fix a collar neighborhood $\mathcal{N}(\partial\Sigma) = (-1, 0] \times \partial\Sigma \subset \Sigma$ and a corresponding collar neighborhood $E|_{\mathcal{N}(\partial\Sigma)} =: \mathcal{N}(\partial_v E) = (-1, 0] \times \partial_v E \subset E$ such that

$$\Pi|_{\mathcal{N}(\partial_v E)} : \mathcal{N}(\partial_v E) \rightarrow \mathcal{N}(\partial\Sigma) : (s, p) \mapsto (s, \Pi(p)).$$

Fix also a collar $\mathcal{N}(\partial_h E) = (-1, 0] \times \partial_h E \subset E$ such that

$$\Pi|_{\mathcal{N}(\partial_h E)} : \mathcal{N}(\partial_h E) \rightarrow \Sigma : (t, p) \mapsto \Pi(p).$$

The intersection of these two collars is then a neighborhood of the corner

$$\mathcal{N}(\partial_v E \cap \partial_h E) := \mathcal{N}(\partial_h E) \cap \mathcal{N}(\partial_v E) = (-1, 0] \times (-1, 0] \times (\partial_h E \cap \partial_v E),$$

and in coordinates (s, t, p) on this neighborhood we have $\Pi(s, t, p) = (s, \Pi(p))$. Given a constant $\epsilon \in (0, 1)$, choose a pair of smooth functions $f_\epsilon, g_\epsilon : (-1, 1) \rightarrow (-1, 1)$ satisfying the following conditions:

- For $\tau \leq -\epsilon$, $f_\epsilon(\tau) = \tau$ and $g_\epsilon(\tau) = 0$,
- For $\tau \in (-\epsilon, \epsilon)$, $f'_\epsilon(\tau) > 0$ and $g'_\epsilon(\tau) < 0$,
- For $\tau \geq \epsilon$, $f_\epsilon(\tau) = 0$ and $g_\epsilon(\tau) = -\tau$.

Denote by $\gamma_\epsilon \subset (-1, 0] \times (-1, 0]$ the image of the smooth path $(f_\epsilon(\tau), g_\epsilon(\tau))$ for $\tau \in (-1, 1)$; this divides $(-1, 0] \times (-1, 0]$ into two connected components. We shall denote the component of $((-1, 0] \times (-1, 0]) \setminus \gamma_\epsilon$ containing $(0, 0)$ by Γ_ϵ (see Figure 2), and then define the compact domain

$$W_\epsilon = E \setminus (\Gamma_\epsilon \times \mathcal{N}(\partial_v E \cap \partial_h E)).$$

This is a smooth manifold with boundary $M_\epsilon := \partial W_\epsilon$, and the latter can be identified with ∂E canonically up to a continuous isotopy which is smooth outside the corner.

Proposition 2.28. *Suppose $\omega \in \Omega_{\text{weak}}(\Pi)$, $\Omega_{\text{strong}}(\Pi)$ or $\Omega_{\text{exact}}(\Pi)$, or $\omega = -d(df \circ J)$ for some $(J, f) \in \mathcal{J}_{\text{Stein}}(\Pi)$. Then for sufficiently small $\epsilon > 0$, the domain W_ϵ with its symplectic or almost Stein data is a weak, strong, exact or almost Stein filling respectively of $(M_\epsilon, \xi_\epsilon)$, where ξ_ϵ is a contact structure supported by a spinal open book with smooth overlap that is isotopic (in the sense of Remark 1.6) to $\partial\Pi$. Moreover, any two fillings obtained in this way by different choices of smoothing are deformation equivalent.*

Proof. Assume $\omega \in \Omega_{\text{weak}}(\Pi)$, so $\omega = d\lambda$ near $\partial_h E$ for some $\lambda \in \Lambda(\partial_h \Pi)$. One can extend λ to a neighborhood of ∂E so that $\lambda \in \Lambda^{\text{fib}}(\partial E)$; this follows from Proposition 2.25 (or a simpler variant focusing only on a neighborhood of $\partial_v E$). Choosing a Liouville form σ on Σ , Proposition 2.13 then implies that for sufficiently large constants $K > 0$, the 1-form

$$\lambda_K := \lambda + K \Pi^* \sigma$$

defines a Giroux form near ∂E . Further, we claim that if $K > 0$ is sufficiently large, then $\lambda_K \wedge \omega > 0$ restricts positively to both $\partial_h E$ and $\partial_v E$. On $\partial_h E$ this is immediate since $\omega = d\lambda$ near $\partial_h E$ and $\lambda|_{T(\partial_h E)}$ is contact, hence

$$\lambda_K \wedge \omega|_{T(\partial_h E)} = (\lambda + K \Pi^* \sigma) \wedge d\lambda|_{T(\partial_h E)} = \lambda \wedge d\lambda|_{T(\partial_h E)} > 0,$$

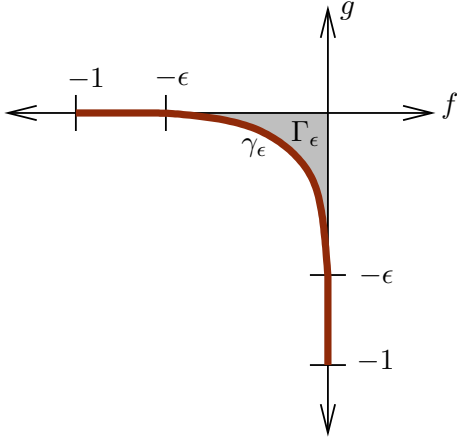


FIGURE 2. The path γ_ϵ and region $\Gamma_\epsilon \subset (-1, 0] \times (-1, 0]$ used for smoothing corners.

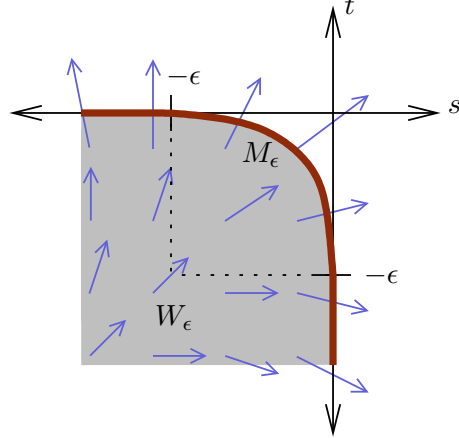


FIGURE 3. The smoothed corner of $M_\epsilon = \partial W_\epsilon$ with the transverse vector field V_K .

where the term $(\Pi^*\sigma \wedge d\lambda)|_{T(\partial_h E)}$ vanishes because both $\Pi^*\sigma$ and $d\lambda|_{T(\partial_h E)}$ kill $V(\partial_h E)$. On $\partial_v E$, we have

$$\lambda_K \wedge \omega|_{T(\partial_v E)} = (\lambda + K \Pi^*\sigma) \wedge \omega|_{T(\partial_v E)} = K \Pi^*\sigma \wedge \omega|_{T(\partial_v E)} + \lambda \wedge \omega|_{T(\partial_v E)},$$

in which the first term is positive since ω is positive on fibers, hence the sum is positive for $K \gg 0$.

Since ω is symplectic, there is a vector field V_K defined near ∂E by the condition $\omega(V_K, \cdot) = \lambda_K$, and $\lambda_K \wedge \omega$ is then positive on any given oriented hypersurface if and only if V_K is positively transverse to that hypersurface. It follows that V_K is everywhere positively transverse to both $\partial_h E$ and $\partial_v E$, so if $\epsilon > 0$ is chosen sufficiently small, then V_K has positive ∂_s and ∂_t components (in the coordinates (s, t, p)) everywhere on

$$(-\epsilon, 0] \times (-\epsilon, 0] \times (\partial_h E \cap \partial_v E) \subset \mathcal{N}(\partial_v E \cap \partial_h E).$$

For this choice of ϵ , V_K is then positively transverse to ∂W_ϵ everywhere (Figure 3), and it follows that

$$\lambda_K \wedge \omega|_{TM_\epsilon} > 0.$$

Thus (W_ϵ, ω) is a weak filling of $(M_\epsilon, \xi_\epsilon)$, where $\xi_\epsilon := \ker(\lambda_K|_{TM_\epsilon})$.

To see that this filling is unique up to symplectic deformation, note first that by the results of the previous subsection, $\omega \in \Omega_{\text{weak}}(\Pi)$ is unique up to homotopy through $\Omega_{\text{weak}}(\Pi)$. Given any such homotopy $\omega_\tau \in \Omega_{\text{weak}}(\Pi)$, $\tau \in [0, 1]$, one can choose a continuous family of primitives $\lambda_\tau \in \Lambda(\partial_h \Pi)$, then extend these to $\lambda_\tau \in \Lambda^{\text{fib}}(\partial \Pi)$ and choose $K > 0$ large enough so that $\lambda_\tau + K \Pi^*\sigma$ defines a continuous family of Giroux forms near ∂E with $(\lambda_\tau + K \Pi^*\sigma) \wedge \omega_\tau$ positive on both $\partial_h E$ and $\partial_v E$. Then for some continuous deformation of the parameter $\epsilon_\tau > 0$, shrinking it as small as necessary for $\tau \in (0, 1)$, we can arrange for $(W_{\epsilon_\tau}, \omega_\tau)$ to be a weak filling of $(M_{\epsilon_\tau}, \ker(\lambda_\tau + K \Pi^*\sigma))$ for all $\tau \in [0, 1]$.

The corresponding statements for strong, exact or almost Stein fillings are proved by a simplification of the above arguments: if ω is strongly convex, we may assume $\omega = d\lambda$ with $\lambda \in \Lambda(\partial \Pi)$, thus $\lambda \wedge d\lambda$ is positive on both boundary faces and the corresponding Liouville vector field plays the role that V_K played above.

It remains to show that the contact structure induced on M_ϵ is supported by a spinal open book isotopic to $\partial\Pi$. It will suffice to show this for a particular choice of $\omega \in \Omega_{\text{strong}}(\Pi)$. Choose a coordinate $\phi \in S^1$ for each connected component of $\partial\Sigma$, so the collar $\mathcal{N}(\partial\Sigma)$ can be viewed as a disjoint union of components $(-1, 0] \times S^1$ with coordinates (s, ϕ) , and we can choose $\sigma = e^s d\phi$ in these collars. Choose also a trivialization of the S^1 -bundle $\partial_h E \cap \partial_v E \rightarrow \partial\Sigma$ and denote the fiber coordinate by $\theta \in S^1$, so each component of $\partial_h E \cap \partial_v E$ now has coordinates $(\phi, \theta) \in T^2$, and the components of $\mathcal{N}(\partial_v E \cap \partial_h E)$ inherit coordinates $(s, t, \phi, \theta) \in (-1, 0] \times (-1, 0] \times T^2$ with

$$\Pi(s, t, \phi, \theta) = (s, \phi).$$

One can then construct a fiberwise Giroux form λ near ∂E that takes the form $e^t d\theta$ in $\mathcal{N}(\partial_v E \cap \partial_h E)$, and extend $d\lambda$ by Proposition 2.25 to a fiberwise symplectic structure $\omega \in \Omega_{\text{strong}}^{\text{fib}}(\Pi)$. Applying Proposition 2.27, this yields a supported strongly convex symplectic structure with a primitive of the form

$$\lambda_K := e^t d\theta + K e^s d\phi$$

on $\mathcal{N}(\partial_v E \cap \partial_h E) = (-1, 0] \times (-1, 0] \times S^1 \times S^1$. Defining W_ϵ as above for any choice of $\epsilon \in (0, 1)$, $M := \partial W_\epsilon$ inherits a spinal open book defined as follows: the spine M_Σ is the complement in $\partial_h E$ of the region $(-\epsilon, 0] \times \{0\} \times T^2 \subset \mathcal{N}(\partial_v E \cap \partial_h E)$, with the fibration

$$\pi_\Sigma : M_\Sigma \rightarrow \Sigma'$$

induced by Π , where Σ' is the complement of the collars $(-\epsilon, 0] \times S^1$ in Σ . The closure of $M \setminus M_\Sigma$ then constitutes the paper M_P , with the ϕ -coordinate defining the fibration $\pi_P : M_P \rightarrow S^1$. The restriction of λ_K to M is now a Giroux form for this spinal open book. \square

3. A CRITERION FOR THE CANONICAL STEIN HOMOTOPY TYPE

The characterization of supported almost Stein structures given in Definition 1.23 is natural, but not general enough to be useful in classifying fillings up to Stein homotopy. In particular, the proof of Theorem B stated in the introduction will require us to consider bordered Lefschetz fibrations $\Pi : E \rightarrow \Sigma$ with almost Stein structures (J, f) for which the fibers are almost complex submanifolds but the projection Π is not pseudoholomorphic. The more general characterization given by Theorem C will therefore be useful, and it can be restated as follows.

Theorem 3.1. *Suppose $\Pi : E \rightarrow \Sigma$ is an allowable bordered Lefschetz fibration, j is a complex structure on Σ and (J, f) is an almost Stein structure on E with the following properties:*

- (1) J restricts to a positively oriented complex structure on the smooth part of every fiber;
- (2) f is constant on the boundary components of every fiber;
- (3) The restriction of $-df \circ J$ to ∂E is a Giroux form for $\partial\Pi$ (cf. Remark 1.13).
- (4) There exists an open neighborhood $\mathcal{U} \subset \Sigma$ of $\partial\Sigma$ such that the map

$$(E|_{\mathcal{U}}, J) \xrightarrow{\Pi} (\mathcal{U}, j)$$

is pseudoholomorphic;

- (5) The maximal J -complex subbundle $\xi_J \subset T(\partial_h E)$ is preserved under the Reeb flow defined via $-df \circ J|_{T(\partial_h E)}$, and $J|_{\xi_J} = \Pi^* j$.

Then (J, f) is almost Stein homotopic to an almost Stein structure supported by Π .

Remark 3.2. We will not use this fact, but one can show that whenever the first and fifth conditions in Theorem 3.1 hold, the projection Π is pseudoholomorphic (for a suitable choice of complex structure on the base) whenever the Nijenhuis tensor takes values in the vertical subbundle. In particular, this is always true if J is integrable.

We begin by generalizing the space $\mathcal{J}(\Pi)$ from Definition 2.15.

Definition 3.3. Given an open subset $\mathcal{U} \subset \Sigma$, let $\mathcal{J}(\Pi; \mathcal{U})$ denote the space of pairs (J, ∂_θ) , where J is an almost complex structure on E defining the correct orientation, ∂_θ is a positively oriented nowhere zero vertical vector field on $\partial_h E$, and Σ admits a complex structure j so that the following conditions are satisfied:

- (1) J restricts to a positively oriented complex structure on the smooth part of every fiber;
- (2) The equation $T\Pi \circ J = j \circ T\Pi$ is satisfied in $E|_{\mathcal{U}}$ and along $\partial_h E$;
- (3) The flow of ∂_θ on $\partial_h E$ is 1-periodic and preserves $\xi_J := T(\partial_h E) \cap JT(\partial_h E)$.

Observe that $\mathcal{J}(\Pi; \Sigma) = \mathcal{J}(\Pi)$, and $\mathcal{J}(\Pi; \mathcal{U}') \subset \mathcal{J}(\Pi; \mathcal{U})$ whenever $\mathcal{U} \subset \mathcal{U}'$; moreover, $\mathcal{J}(\Pi; \mathcal{U})$ is contractible for every choice of $\mathcal{U} \subset \Sigma$. Theorem 3.1 would thus follow immediately if we could show that every $(J, \partial_\theta) \in \mathcal{J}(\Pi; \mathcal{U})$ admits a suitable J -convex function, but this is probably not true in general—we at least have been unable to prove it except when $\mathcal{U} = \Sigma$. What we will show instead is that if \mathcal{U} contains $\partial\Sigma$, then every $(J, \partial_\theta) \in \mathcal{J}(\Pi; \mathcal{U})$ has a *perturbation* that admits a suitable J -convex function, and this perturbation can be arranged to depend continuously on parameters. Here is the more technical result that implies Theorem 3.1:

Proposition 3.4. *Assume $\Pi : E \rightarrow \Sigma$ is allowable, $\mathcal{U} \subset \Sigma$ is an open neighborhood of $\partial\Sigma$, X is a compact cell complex, $A \subset X$ is a subcomplex, and*

$$\begin{aligned} X &\rightarrow \mathcal{J}(\Pi; \mathcal{U}) : \tau \mapsto (J_\tau, \partial_\theta^\tau), \\ A &\rightarrow C^\infty(E) : \tau \mapsto f_\tau \end{aligned}$$

are continuous maps such that for every $\tau \in A$, (J_τ, f_τ) is an almost Stein structure, f_τ is constant on all boundary components of fibers, and $\lambda_\tau := -df_\tau \circ J_\tau$ restricts to ∂E as a Giroux form for $\partial\Pi$ (in the sense of Remark 1.13) satisfying $\lambda_\tau(\partial_\theta^\tau) \equiv \text{const}$. Then there exists a continuous (with respect to the C^∞ -topology) family of almost Stein structures $\{(J'_\tau, f'_\tau)\}_{\tau \in X}$ matching (J_τ, f_τ) for all $\tau \in A$ such that J'_τ is C^∞ -close to J_τ for all $\tau \in X$.

The proof of Proposition 3.4 requires several steps and will occupy the remainder of this section, so here is an initial sketch. Let $\{j_\tau\}_{\tau \in X}$ denote the uniquely determined family of complex structures on Σ such that $\Pi : (E, J_\tau) \rightarrow (\Sigma, j_\tau)$ is holomorphic in $E|_{\mathcal{U}}$ and along $\partial_h E$ for all τ . Since \mathcal{U} is open, we can choose a function $\varphi : \Sigma \rightarrow \mathbb{R}$ which has all its critical points in \mathcal{U} and is j_τ -convex for every τ . Holomorphicity of Π then allows the construction of J_τ -convex functions on $E|_{\mathcal{U}}$ using the Thurston trick as in Prop. 2.6. Outside of $E|_{\mathcal{U}}$, the function $\varphi \circ \Pi$ has level sets that are unions of J_τ -holomorphic fibers and are thus Levi-flat, i.e. the maximal complex subbundle in each level set is a foliation. A suitable choice of fiberwise Liouville structure then allows us to perturb these foliations to contact structures as in the Thurston-Winkelkemper construction [TW75] of contact forms supported by open books (cf. Prop. 2.1). The almost complex structures admit corresponding perturbations J'_τ that preserve these contact structures, so that the function $\varphi \circ \Pi$, after modifying φ to make φ'' sufficiently large, becomes J'_τ -convex. This makes use of Lemma 2.21, and it produces J'_τ -convex functions f'_τ that match $\varphi \circ \Pi$ away from $E|_{\text{Crit}(\varphi)}$ and take the form $\varphi \circ \Pi + \epsilon f_\tau$ near

$E|_{\text{Crit}(\varphi)}$. Actually proving that (J'_τ, f'_τ) are almost Stein structures requires also showing that the Liouville forms $-df'_\tau \circ J'_\tau$ restrict to the smooth faces of ∂E as contact forms. Moreover, we need to be able to keep this condition under linear interpolations between our constructed functions f'_τ and the original f_τ in order solve the extension problem. Both steps will make essential use of the holomorphicity of Π at ∂E , as well as the Thurston trick: a crucial detail for the latter is that the original family of J_τ -convex functions $\{f_\tau\}_{\tau \in A}$ can easily be extended to $\tau \in X$ as a family of *fiberwise* J_τ -convex functions, which we use in the construction of J'_τ and f'_τ .

We now proceed with the details of the argument sketched above. As in the statement of Proposition 3.4, all families of objects parametrized by X will be assumed in the following to be continuous in the C^∞ -topology, and $\mathcal{U} \subset \Sigma$ will be an open neighborhood of $\partial\Sigma$. We will sometimes find it convenient to replace \mathcal{U} with a smaller neighborhood of $\partial\Sigma$, which is not a loss of generality since it enlarges the space $\mathcal{J}(\Pi; \mathcal{U})$. In particular, since all critical values of Π are in the interior, let us start by assuming

$$\bar{\mathcal{U}} \cap \Sigma^{\text{crit}} = \emptyset.$$

3.1. Weinstein structures on the base.

Lemma 3.5. *There exists a smooth function $\varphi : \Sigma \rightarrow \mathbb{R}$ which is j_τ -convex for all τ and constant on $\partial\Sigma$, and has all its critical points in $\mathcal{U} \setminus \partial\Sigma$.*

Proof. Start by choosing a Morse function $\varphi : \Sigma \rightarrow \mathbb{R}$ that is regular and constant on the boundary and has no local maxima. By composing with a suitable diffeomorphism of Σ , we can arrange that $\text{Crit}(\varphi) \subset \mathcal{U} \setminus \partial\Sigma$. Now since every critical point has Morse index 0 or 1, we can fix local coordinates (x, y) near each critical point so that, up to addition of constants, $\varphi(x, y)$ takes the form $x^2 + y^2$ or $x^2 - y^2$. Given any constant $c > 0$, we can further modify φ by composing with a diffeomorphism supported near the index 1 critical points so that these (in the same coordinates!) now take the form $cx^2 - y^2$. Since the parameter space X is compact, Lemma 3.6 below now implies that by selecting c sufficiently large, we can assume φ is j_τ -convex near $\text{Crit}(\varphi)$ for all $\tau \in X$. Lemma 2.21 (with Remark 2.22) can then be applied to make φ into a globally j_τ -convex function for all $\tau \in X$ by postcomposing it with a sufficiently convex function $\mathbb{R} \rightarrow \mathbb{R}$. \square

The above proof required the following lemma:

Lemma 3.6. *Suppose j is a smooth almost complex structure on a neighborhood of 0 in \mathbb{C} , compatible with the canonical orientation, and let $\varphi_0, \varphi_1 : \mathbb{C} \rightarrow \mathbb{R}$ denote the functions*

$$\varphi_0(x + iy) = x^2 + y^2, \quad \varphi_1(x + iy) = cx^2 - y^2,$$

where $c > 0$ is a constant. Then φ_0 is j -convex near 0, and φ_1 is also j -convex near 0 whenever c is sufficiently large.

Proof. Let j_0 denote the ‘‘constant’’ complex structure on \mathbb{C} that matches j at the origin, in other words $j_0(z) := j(0)$ for all $z \in \mathbb{C}$. We claim first that the statement of the lemma is true if j is replaced by j_0 . Indeed, j_0 can be written as the matrix

$$j_0 = \begin{pmatrix} a & -\frac{1+a^2}{b} \\ b & -a \end{pmatrix},$$

where a and b are real constants with $b > 0$ (due to the orientation assumption). Then we compute:

$$\begin{aligned} -d(d\varphi_0 \circ j_0) &= 2 \left(\frac{1+a^2}{b} + b \right) dx \wedge dy, \\ -d(d\varphi_1 \circ j_0) &= 2 \left(c \frac{1+a^2}{b} - b \right) dx \wedge dy. \end{aligned}$$

The first is always positive, and the second is positive if and only if $c > b^2/(1+a^2)$, so this proves the claim about j_0 . To generalize this to j , it suffices to observe that since $d\varphi_0(0) = d\varphi_1(0) = 0$, the 1-jets of $-d\varphi_0 \circ j$ and $-d\varphi_1 \circ j$ at 0 (and hence also the question of j -convexity on some neighborhood of that point) depend on $j(0)$ but not on the derivatives of j , so the fact that $j(0) = j_0(0)$ implies the result. \square

For the remainder of this section, we fix a function $\varphi : \Sigma \rightarrow \mathbb{R}$ as given by Lemma 3.5 and define the family of 1-forms

$$\sigma_\tau = -d\varphi \circ j_\tau.$$

By construction, $d\sigma_\tau > 0$ everywhere and $d\varphi \wedge \sigma_\tau > 0$ away from $\text{Crit}(\varphi)$, for all $\tau \in X$. In particular, this means that $d\sigma_\tau$, together with φ and the family of Liouville vector fields $d\sigma_\tau$ -dual to σ_τ , define a family of Weinstein structures on Σ .

3.2. Perturbing J near Lefschetz critical points. Define the function

$$F = \varphi \circ \Pi : E \rightarrow \mathbb{R}.$$

We shall now define a family of perturbations of J_τ near E^{crit} that make F plurisubharmonic on this neighborhood. For any $p \in E^{\text{crit}}$, let $\mathcal{N}(p)$ denote an open neighborhood of p , which we will always assume is arbitrarily small in order to satisfy various conditions. The first such condition is that $\mathcal{N}(p)$ admits complex coordinates (z_1, z_2) identifying p with $(0, 0) \in \mathbb{C}^2$ so that $\Pi(z_1, z_2) = z_1^2 + z_2^2$ for a suitable choice of complex coordinate z on a neighborhood $\mathcal{N}(\Pi(p)) \subset \Sigma$ of $\Pi(p)$, identifying $\Pi(p)$ with $0 \in \mathbb{C}$. We shall abbreviate the pair of coordinates on $\mathcal{N}(p)$ together as $\zeta = (z_1, z_2)$, and write the real and imaginary parts as

$$\zeta = (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in \mathcal{N}(p), \quad z = x + iy \in \mathcal{N}(\Pi(p)).$$

Note that the formula for $\Pi(z_1, z_2)$ is invariant under simultaneous coordinate changes of the form

$$(z_1, z_2) \mapsto (az_1, az_2), \quad z \mapsto a^2 z$$

for any $a \in \mathbb{C}$, thus we can choose a suitable constant a and make such a transformation such that without loss of generality, the local coordinate expression for φ near $\Pi(p)$ satisfies

$$d\varphi(0) = dx.$$

This is possible because we have already arranged for all critical points of φ to be separate from Σ^{crit} ; indeed, $\text{Crit}(\varphi) \subset \mathcal{U}$ and $\mathcal{U} \cap \Sigma^{\text{crit}} = \emptyset$, where the latter can always be achieved by making \mathcal{U} a smaller neighborhood of $\partial\Sigma$. In particular, φ then has the same 1-jet at $\Pi(p)$ as the locally defined function

$$\varphi_0(x + iy) := x + \varphi(0).$$

We shall repeatedly make use of this fact via the following lemma, which is an easy consequence of the fact that $d\varphi_0(0) = d\varphi(0)$ and $d\Pi(0, 0) = 0$.

Lemma 3.7. *The functions $F = \varphi \circ \Pi : E \rightarrow \mathbb{R}$ and $F_0 := \varphi_0 \circ \Pi : \mathcal{N}(p) \rightarrow \mathbb{R}$ have the same 2-jet at p . Moreover, for any smooth bundle endomorphism $A : \mathcal{N}(p) \rightarrow \text{End}(TE|_{\mathcal{N}(p)})$, the 1-forms $dF \circ A$ and $dF_0 \circ A$ have matching 1-jets at p , which depend on $A(p)$ but not on the derivatives of A at p . \square*

Denote by i the standard complex structure on \mathbb{C}^2 and identify this with an integrable complex structure on $\mathcal{N}(p)$ via the coordinates (z_1, z_2) . For any i -antilinear map Y on \mathbb{C}^2 sufficiently close to 0, one can define another complex structure close to i by

$$\Phi(Y) := \left(\mathbf{1} + \frac{1}{2}iY \right) i \left(\mathbf{1} + \frac{1}{2}iY \right)^{-1}.$$

Indeed, Φ can be regarded as the inverse of a local chart for the manifold of complex structures $\mathcal{J}(\mathbb{C}^2)$, identifying a neighborhood of i in $\mathcal{J}(\mathbb{C}^2)$ with a neighborhood of 0 in $T_i\mathcal{J}(\mathbb{C}^2)$ such that $d\Phi(0)$ is the identity on the space of i -antilinear maps. By Proposition 2.10, $J_\tau(p) = i$ for all $\tau \in X$, thus there is a family of smooth maps $Y_\tau : \mathcal{N}(p) \rightarrow T_i\mathcal{J}(\mathbb{C}^2)$ such that for all $\zeta \in \mathcal{N}(p)$,

$$J_\tau(\zeta) = \Phi(Y_\tau(\zeta)),$$

and $Y_\tau(0) = 0$.

Working in real coordinates (x_1, y_1, x_2, y_2) , define the i -antilinear matrix

$$Y' = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We use this to define for all $\epsilon \geq 0$ sufficiently small a family of perturbed almost complex structures on $\mathcal{N}(p)$ by

$$J_\tau^\epsilon(\zeta) = \Phi(Y_\tau(\zeta) + \epsilon Y').$$

Let

$$Y'_\tau(\zeta) = \left. \frac{\partial}{\partial \epsilon} J_\tau^\epsilon(\zeta) \right|_{\epsilon=0}.$$

Then since $Y_\tau(0, 0) = 0$ and $d\Phi(0)$ is the identity, we have $Y'_\tau(0, 0) = Y'$. Let

$$\Lambda_\tau^\epsilon = -dF \circ J_\tau^\epsilon,$$

and for $\epsilon > 0$, define smooth families of 1-forms $\widehat{\eta}_\tau^\epsilon$ via the formula

$$(3.1) \quad \Lambda_\tau^\epsilon = \Lambda_\tau^0 + \epsilon \widehat{\eta}_\tau^\epsilon.$$

There is a smooth extension of $\widehat{\eta}_\tau^\epsilon$ to $\epsilon = 0$, namely

$$\widehat{\eta}_\tau^0 := \lim_{\epsilon \rightarrow 0} \frac{\Lambda_\tau^\epsilon - \Lambda_\tau^0}{\epsilon} = \left. \frac{\partial}{\partial \epsilon} \Lambda_\tau^\epsilon \right|_{\epsilon=0} = -dF \circ Y'_\tau.$$

Lemma 3.8. *There exists a constant $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$ and $\tau \in X$, $d\Lambda_\tau^\epsilon$ is symplectic on $\mathcal{N}(p)$ and tames both i and J_τ^ϵ . Moreover, $d\widehat{\eta}_\tau^0$ is also symplectic on $\mathcal{N}(p)$ and tames J_τ for all $\tau \in X$.*

Proof. We first prove the claim about $d\widehat{\eta}_\tau^0$, for which it suffices to show that $d\widehat{\eta}_\tau^0|_p$ tames i since $J_\tau(p) = i$ for all $\tau \in X$ and the taming condition is open. Consider the slightly simpler 1-form

$$\widehat{\eta}_0 := -d(\varphi_0 \circ \Pi) \circ Y',$$

where we recall $\varphi_0(x + iy) = x + \varphi(0)$. We then have $\varphi_0 \circ \Pi(z_1, z_2) - \varphi(0) = \operatorname{Re}(z_1^2 + z_2^2) = \sum_{j=1}^2 (x_j^2 - y_j^2)$ and

$$dx_j \circ Y' = -dy_j, \quad dy_j \circ Y' = -dx_j \quad \text{for } j = 1, 2,$$

thus

$$\widehat{\eta}_0 = 2 \sum_{j=1}^2 (x_j dy_j - y_j dx_j),$$

giving $d\widehat{\eta}_0 = 4 \sum_{j=1}^2 dx_j \wedge dy_j$, which clearly tames i . Now Lemma 3.7 implies that $\widehat{\eta}_\tau^0$ and $\widehat{\eta}_0$

have the same 1-jet at p , hence $d\widehat{\eta}_\tau^0|_p = d\widehat{\eta}_0|_p$, and the claim follows.

Next we show that $d\Lambda_\tau^\epsilon$ tames both i and J_τ^ϵ on $\mathcal{N}(p)$ when ϵ is positive but small. Observe first that $d\Lambda_\tau^0|_p = 0$: indeed, by Lemma 3.7 this holds if the 1-form $\lambda := -d(\varphi_0 \circ \Pi) \circ i$ satisfies $d\lambda|_p = 0$, and since $\varphi_0 \circ \Pi(z_1, z_2) = \sum_{j=1}^2 (x_j^2 - y_j^2) + \varphi(0)$, an explicit computation shows

$$\lambda = -d(\varphi_0 \circ \Pi) \circ i = 2 \sum_{j=1}^2 (x_j dy_j + y_j dx_j),$$

which is everywhere closed. It will now suffice to show that for any $\tau \in X$ and any nonzero vector $v \in T_p E$, the derivatives

$$(3.2) \quad \left. \frac{d}{d\epsilon} d\Lambda_\tau^\epsilon(v, iv) \right|_{\epsilon=0}, \quad \left. \frac{d}{d\epsilon} d\Lambda_\tau^\epsilon(v, J_\tau^\epsilon v) \right|_{\epsilon=0}$$

are both positive. Since $\frac{\partial}{\partial \epsilon} \Lambda_\tau^\epsilon|_{\epsilon=0} = \widehat{\eta}_\tau^0$, both of these are equal to $d\widehat{\eta}_\tau^0(v, iv)$, which is positive by the first claim proved above. Note that in computing the expression on the right in (3.2), derivative of J_τ^ϵ with respect to ϵ does not appear since $d\Lambda_\tau^0|_p = 0$. \square

To summarize this step so far, we have defined a family of almost complex structures $\{J_\tau^\epsilon\}_{\epsilon \in [0, \epsilon_0], \tau \in X}$ near E^{crit} such that $J_\tau^0 = J_\tau$ and $F = \varphi \circ \Pi$ is J_τ^ϵ -convex for $\epsilon > 0$. Moreover, the Liouville forms $\Lambda_\tau^\epsilon = -dF \circ J_\tau^\epsilon$ for $\epsilon > 0$ can be written as $\Lambda_\tau^\epsilon = \Lambda_\tau^0 + \epsilon \widehat{\eta}_\tau^\epsilon$, where $\widehat{\eta}_\tau^\epsilon$ is a smooth family of 1-forms that define fiberwise Liouville forms near E^{crit} and converge as $\epsilon \rightarrow 0$ to a fiberwise Liouville form $\widehat{\eta}_\tau^0$ such that $d\widehat{\eta}_\tau^0$ tames both i and J_τ . The main point of this construction was that it gives rise to a family of contact structures on the level sets of F : indeed, define on $\mathcal{N}(p) \setminus \{p\}$ a family of co-oriented 2-plane distributions

$$\xi_\tau^\epsilon := \ker dF \cap \ker \Lambda_\tau^\epsilon.$$

These are J_τ^ϵ -invariant, so the fact that F is J_τ^ϵ -convex for $\epsilon > 0$ implies that they are contact on each level set of F whenever $\epsilon > 0$. For $\epsilon = 0$ this is not the case, as

$$\xi_\tau^0 = VE$$

is the vertical subbundle of the Lefschetz fibration and thus defines foliations on the level sets of F . In the next step, we will use the Liouville forms σ_τ from §3.1 to extend ξ_τ^ϵ over the rest of $E \setminus (E^{\operatorname{crit}} \cup E|_{\operatorname{Crit}(\varphi)})$. To do this we will need Lemma 3.11 below, for which the next two lemmas are preparation. In the following, we use the coordinates $\zeta = (z_1, z_2)$ to define Euclidean norms $|v|$ of vectors $v \in TE|_{\mathcal{N}(p)}$, and keep in mind that $\mathcal{N}(p)$ can always be made smaller if necessary.

Lemma 3.9. *There exists a constant $c_1 > 0$ and a family of smooth vector fields R_τ on $\mathcal{N}(p) \setminus \{p\}$ such that $|R_\tau| \equiv 1$, $dF(R_\tau) \equiv 0$, and*

$$\Lambda_\tau^0(R_\tau(\zeta)) \geq c_1 |\zeta| \quad \text{for all } \zeta \in \mathcal{N}(p) \setminus \{p\}.$$

Proof. Choose a family of J_τ -invariant Riemannian metrics g_τ on $\mathcal{N}(p)$ and let $\nabla^\tau F$ denote the corresponding gradient vector fields of F . Note that since τ lives in a compact parameter space, the norms defined via g_τ are uniformly (with respect to τ) equivalent to the Euclidean norm. By Lemma 3.7, the Hessian of F at p matches that of $\varphi_0 \circ \Pi(\zeta) = \sum_{j=1}^2 (x_j^2 - y_j^2) + \varphi(0)$, thus the critical point of F at $\zeta = 0$ is nondegenerate. It follows that one can find a constant $k > 0$ such that

$$|\nabla^\tau F(\zeta)| \geq k|\zeta| \quad \text{for all } \zeta \in \mathcal{N}(p), \tau \in X.$$

A family of vector fields with the desired properties can then be defined by $R_\tau = \frac{J_\tau \nabla^\tau F}{|J_\tau \nabla^\tau F|}$. \square

Lemma 3.10. *On $\mathcal{N}(p) \setminus \{p\}$, $d\Lambda_\tau^0|_{\xi_\tau^0} = 0$, $\Lambda_\tau^0 \wedge d\Lambda_\tau^0|_{\ker dF} = 0$, and*

$$\left. \frac{\partial}{\partial \epsilon} (\Lambda_\tau^\epsilon \wedge d\Lambda_\tau^\epsilon|_{\ker dF}) \right|_{\epsilon=0} > 0.$$

Proof. Since $\ker(\Lambda_\tau^0|_{\ker dF}) = \xi_\tau^0 = VE$, the first two statements are both equivalent to the fact that VE defines a foliation on every level set of F . We will now prove that the third claim holds after shrinking the neighborhood $\mathcal{N}(p)$ sufficiently. Recall from the proof of Lemma 3.8 that $d\Lambda_\tau^0|_p = 0$, and similarly, $\hat{\eta}_\tau^0 = -dF \circ Y'_\tau$ vanishes at p . Since both are smooth, this implies there is a constant $c_2 > 0$ such that

$$(3.3) \quad \|d\Lambda_\tau^0|_\zeta\| \leq c_2 |\zeta|, \quad \|\hat{\eta}_\tau^0|_\zeta\| \leq c_2 |\zeta| \quad \text{for } \zeta \in \mathcal{N}(p),$$

where we denote by $\|\cdot\|$ the natural norm induced on tensors from the Euclidean norm in the coordinates. Now for any $\zeta \in \mathcal{N}(p) \setminus \{p\}$, fix $v \in T_\zeta E$ with $v \in VE$ and $|v| = 1$, so the vector $iv \in T_\zeta E$ is also vertical and also has norm 1. Denote the value at ζ of the vector field from Lemma 3.9 by $R := R_\tau(\zeta) \in \ker dF|_\zeta$, which according to the lemma, satisfies

$$(3.4) \quad \Lambda_\tau^0(R) \geq c_1 |\zeta|$$

for some constant $c_1 > 0$ independent of ζ and τ . The triple (R, v, iv) now form a positively oriented basis of $\ker dF|_\zeta$, and $\Lambda_\tau^\epsilon \wedge d\Lambda_\tau^\epsilon(R, v, iv)$ is proportional to

$$\Lambda_\tau^\epsilon(R) d\Lambda_\tau^\epsilon(v, iv) + \Lambda_\tau^\epsilon(v) d\Lambda_\tau^\epsilon(iv, R) + \Lambda_\tau^\epsilon(iv) d\Lambda_\tau^\epsilon(R, v).$$

Differentiating this with respect to ϵ and setting $\epsilon = 0$, three terms drop out since $\Lambda_\tau^0(v) = \Lambda_\tau^0(iv) = d\Lambda_\tau^0(v, iv) = 0$, and we are left with

$$\begin{aligned} \Lambda_\tau^0(R) d\hat{\eta}_\tau^0(v, iv) + \hat{\eta}_\tau^0(v) d\Lambda_\tau^0(iv, R) + \hat{\eta}_\tau^0(iv) d\Lambda_\tau^0(R, v) \\ \geq c_1 |\zeta| d\hat{\eta}_\tau^0(v, iv) - 2c_2^2 |\zeta|^2 = |\zeta| \cdot (c_1 d\hat{\eta}_\tau^0(v, iv) - 2c_2^2 |\zeta|), \end{aligned}$$

where we've used (3.3) to bound the magnitude of the last two terms from above and (3.4) to bound the first from below. Since $d\hat{\eta}_\tau^0$ tames i and $|v| = 1$, the term $d\hat{\eta}_\tau^0(v, iv)$ satisfies a uniform positive lower bound on $\mathcal{N}(p)$, thus the entire expression becomes positive as soon as $|\zeta|$ is sufficiently small. \square

Lemma 3.11. *There exists a family of smooth 1-forms η_τ^ϵ on $\mathcal{N}(p) \setminus \{p\}$, for $\tau \in X$ and $\epsilon \in [0, \epsilon_0]$, such that $d\eta_\tau^0|_{VE} > 0$ and*

$$\xi_\tau^\epsilon = \ker dF \cap \ker(\Pi^* \sigma_\tau + \epsilon \eta_\tau^\epsilon).$$

Moreover, the η_τ^ϵ decay (uniformly in τ and ϵ) to zero at p .

Proof. On $\mathcal{N}(p) \setminus \{p\}$, the kernels of Λ_τ^0 and $\Pi^* \sigma_\tau$ restricted to level sets of F are both $\xi_\tau^0 = VE$, thus there is a family of smooth positive functions $g_\tau : \mathcal{N}(p) \setminus \{p\} \rightarrow (0, \infty)$ such that

$$(3.5) \quad \Pi^* \sigma_\tau|_{\ker dF} = g_\tau \Lambda_\tau^0|_{\ker dF}.$$

We can plug in the unit vector field R_τ from Lemma 3.9 to compute g_τ in coordinates, and since $\Pi^* \sigma_\tau$ vanishes at p , the estimate in the lemma gives rise to an estimate

$$|g_\tau(\zeta)| = \frac{|\Pi^* \sigma_\tau(R_\tau(\zeta))|}{|\Lambda_\tau^0(R_\tau(\zeta))|} \leq \frac{c_2 |\zeta|}{c_1 |\zeta|} = \frac{c_2}{c_1}$$

for some constants $c_1, c_2 > 0$, so that the functions g_τ are uniformly bounded near p .

The relation (3.5) together with (3.1) now implies

$$g_\tau \Lambda_\tau^\epsilon|_{\ker dF} = (\Pi^* \sigma_\tau + \epsilon g_\tau \hat{\eta}_\tau^\epsilon)|_{\ker dF},$$

thus we can set

$$\eta_\tau^\epsilon := g_\tau \hat{\eta}_\tau^\epsilon \quad \text{for } \epsilon \in [0, \epsilon_0], \tau \in X.$$

Observe that $\hat{\eta}_\tau^\epsilon|_p = 0$ for all τ and ϵ by definition, so the boundedness of g_τ implies that η_τ^ϵ also decays uniformly to zero at p .

Our remaining task is to show that $d\eta_\tau^0|_{VE} > 0$ in some (possibly smaller) open set of the form $\mathcal{N}(p) \setminus \{p\}$. Since $g_\tau \Lambda_\tau^\epsilon \wedge d(g_\tau \Lambda_\tau^\epsilon)|_{\ker dF} = g_\tau^2 \Lambda_\tau^\epsilon \wedge d\Lambda_\tau^\epsilon|_{\ker dF}$, Lemma 3.10 implies that on a sufficiently small neighborhood $\mathcal{N}(p) \setminus \{p\}$,

$$\begin{aligned} 0 &< g_\tau^2 \frac{\partial}{\partial \epsilon} (\Lambda_\tau^\epsilon \wedge d\Lambda_\tau^\epsilon|_{\ker dF}) \Big|_{\epsilon=0} \\ &= \frac{\partial}{\partial \epsilon} [(\Pi^* \sigma_\tau + \epsilon \eta_\tau^\epsilon) \wedge d(\Pi^* \sigma_\tau + \epsilon \eta_\tau^\epsilon)|_{\ker dF}] \Big|_{\epsilon=0} \\ &= \frac{\partial}{\partial \epsilon} [\epsilon \cdot (\Pi^* \sigma_\tau + \epsilon \eta_\tau^\epsilon) \wedge d\eta_\tau^\epsilon|_{\ker dF}] \Big|_{\epsilon=0} = \Pi^* \sigma_\tau \wedge d\eta_\tau^0|_{\ker dF}, \end{aligned}$$

where we've used the fact that $\Pi^* \sigma_\tau$ is closed on the level sets of F since σ_τ is (obviously) closed on the level sets of φ . The kernel of $\Pi^* \sigma|_{\ker dF}$ is VE , so this last relation is equivalent to $d\eta_\tau^0|_{VE} > 0$. \square

3.3. Perturbing from Levi flat to contact. By assumption, there is a subcomplex $A \subset X$ and a family of smooth functions $\{f_\tau : E \rightarrow \mathbb{R}\}_{\tau \in A}$ such that $\lambda_\tau := -df_\tau \circ J_\tau$ are Liouville forms and restrict to ∂E as Giroux forms in the sense of Remark 1.13. Recall from Definition 2.17 the convex space $PSH_{(J, \partial_\theta)}^{\text{fib}}(\Pi)$ of fiberwise J -convex functions associated to each $(J, \partial_\theta) \in \mathcal{J}(\Pi)$. It will be convenient to observe that this definition still makes sense and $PSH_{(J, \partial_\theta)}^{\text{fib}}(\Pi)$ is still convex if (J, ∂_θ) is only assumed to belong to $\mathcal{J}(\Pi; \mathcal{U})$. For example, $f_\tau \in PSH_{(J_\tau, \partial_{\theta_\tau}^\tau)}^{\text{fib}}(\Pi)$ for each $\tau \in A$.

Lemma 3.12. *The family of functions $\{f_\tau : E \rightarrow \mathbb{R}\}_{\tau \in A}$ can be extended to a family $\{f_\tau : E \rightarrow \mathbb{R}\}_{\tau \in X}$ such that $f_\tau \in PSH_{(J_\tau, \partial_{\theta_\tau}^\tau)}^{\text{fib}}(\Pi)$ for every $\tau \in X$.*

Proof. Independently of the given functions f_τ , we first observe that there exists a family $\{g_\tau \in PSH_{(J_\tau, \partial_{\theta_\tau}^\tau)}^{\text{fib}}(\Pi)\}_{\tau \in X}$. In light of Remark 2.26, this follows from the partition of unity argument in the proof of Proposition 2.25; the only meaningful difference is that one needs to consider families depending continuously on τ at every step, though since X is compact,

one can also use Lemmas 2.21 and 3.6 to construct g_τ so that it is independent of τ away from $\partial_h E$. This establishes the lemma in the case $A = \emptyset$.

To solve the extension problem in general, it suffices to consider the case where X is a disk \mathbb{D}^k and $A = \partial\mathbb{D}^k = S^{k-1}$ for some $k \in \mathbb{N}$. We start by extending the given family $\{f_\tau\}_{\tau \in A}$ arbitrarily to a family of smooth functions $\hat{f}_\tau : E \rightarrow \mathbb{R}$ for $\tau \in X$ such that each $\hat{f}_\tau|_{\partial_h E}$ is invariant under the S^1 -action defined by the flow of ∂_θ^τ and the normal derivatives $d\hat{f}_\tau(-J_\tau \partial_\theta^\tau)$ are locally constant for each τ . Since each J_τ has an S^1 -invariant restriction to $\partial_h E$, the 1-forms $\alpha_\tau := -d\hat{f}_\tau \circ J_\tau|_{T(\partial_h E)}$ are also S^1 -invariant and thus satisfy

$$0 = \mathcal{L}_{\partial_\theta^\tau} \alpha_\tau = d(\alpha_\tau(\partial_\theta^\tau)) + d\alpha_\tau(\partial_\theta^\tau, \cdot).$$

In this expression, the first term at the right vanishes since $\alpha_\tau(\partial_\theta^\tau) = -d\hat{f}_\tau(J_\tau \partial_\theta^\tau)$ is locally constant, thus $d\alpha_\tau(\partial_\theta^\tau, \cdot) \equiv 0$. The remaining conditions in the definition of a fiberwise J_τ -convex function are all open, so it follows that \hat{f}_τ is also fiberwise J_τ -convex for every τ in some open neighborhood $A' \subset X$ of A . Finally, choose a cutoff function $\beta : X \rightarrow [0, 1]$ that is supported in A' and satisfies $\beta|_A \equiv 1$, and set $f_\tau := \beta(\tau)\hat{f}_\tau + [1 - \beta(\tau)]g_\tau$. This is fiberwise J_τ -convex for every $\tau \in X$ since the space $PSH_{(J_\tau, \partial_\theta^\tau)}^{fib}(\Pi)$ is convex. \square

From now on, denote by

$$\lambda_\tau = -df_\tau \circ J_\tau, \quad \tau \in X$$

the family of fiberwise Liouville forms on E that arise from the above lemma. Choose a neighborhood $\mathcal{N}(E^{\text{crit}}) \subset E$ of E^{crit} such that the 1-forms η_τ^ϵ from Lemma 3.11 are defined and satisfy $d\eta_\tau^0|_{VE} > 0$ on an open neighborhood of $\overline{\mathcal{N}(E^{\text{crit}})} \setminus E^{\text{crit}}$. For any smaller neighborhood $\mathcal{N}'(E^{\text{crit}})$ of E^{crit} with compact closure in $\mathcal{N}(E^{\text{crit}})$, we can choose $\epsilon_0 > 0$ small enough so that

$$d\eta_\tau^\epsilon|_{VE} > 0 \quad \text{on } \mathcal{N}(E^{\text{crit}}) \setminus \mathcal{N}'(E^{\text{crit}}) \text{ for all } \epsilon \in [0, \epsilon_0].$$

The following lemma should be understood to be true after a possible further shrinking of the neighborhoods $\mathcal{N}(E^{\text{crit}})$ and $\mathcal{N}'(E^{\text{crit}})$ together with the number $\epsilon_0 > 0$.

Lemma 3.13. *The family of 1-forms η_τ^ϵ on $\mathcal{N}(E^{\text{crit}}) \setminus \mathcal{N}'(E^{\text{crit}})$ for $\tau \in X$ and $\epsilon \in [0, \epsilon_0]$ can be extended over $E \setminus \mathcal{N}'(E^{\text{crit}})$ so that $d\eta_\tau^\epsilon|_{VE} > 0$ everywhere and $\eta_\tau^\epsilon = \lambda_\tau$ near $\partial_h E$ and in $E|_{\mathcal{U}'}$ for some neighborhood $\mathcal{U}' \subset \Sigma$ of $\text{Crit}(\varphi) \cup \partial\Sigma$.*

Proof. We again use a variant of the partition of unity argument from Proposition 2.25. For step 1, choose an open neighborhood $\mathcal{U}' \subset \Sigma$ of $\text{Crit}(\varphi) \cup \partial\Sigma$ with closure in \mathcal{U} , and for each regular value $z \in \Sigma \setminus \mathcal{U}'$ of Π , choose a neighborhood $\mathcal{U}_z \subset \Sigma \setminus \Sigma^{\text{crit}}$ of z together with a family of 1-forms $\eta_{\tau,z}^\epsilon$ on $E|_{\mathcal{U}_z}$ such that $d\eta_{\tau,z}^\epsilon$ is positive on fibers and $\eta_{\tau,z}^\epsilon = \lambda_\tau$ near $\partial_h E$. We can arrange this moreover so that \mathcal{U}_z is disjoint from $\overline{\mathcal{U}'}$ whenever $z \notin \overline{\mathcal{U}'}$ and $\mathcal{U}_z \subset \mathcal{U}$ for $z \in \overline{\mathcal{U}'}$, which permits the choice $\eta_{\tau,z}^\epsilon := \lambda_\tau$ in the latter case.

Step 2 is to define $\eta_{\tau,z}^\epsilon$ on $E|_{\mathcal{U}_z}$ for a neighborhood $\mathcal{U}_z \subset \Sigma$ of each $z \in \Sigma^{\text{crit}}$, matching the given η_τ^ϵ near E^{crit} . We start by extending η_τ^ϵ over each component of $E_z \setminus E_z^{\text{crit}}$ as a Liouville form, which is possible since Π is allowable, though there is a slightly subtle point if we want to arrange $\eta_{\tau,z}^\epsilon = \lambda_\tau$ near $\partial_h E$: Stokes' theorem may make this impossible if there are vanishing cycles $C \subset E_z$ near E^{crit} on which $\int_C \eta_{\tau,z}^\epsilon$ is too large. Recall however that while η_τ^ϵ may fail to be smooth at E^{crit} , it does have a (uniformly in τ and ϵ) continuous extension that vanishes at E^{crit} , so its integrals along cycles in $\mathcal{N}(E^{\text{crit}})$ can be assumed arbitrarily small if we replace $\mathcal{N}(E^{\text{crit}})$ by a suitably smaller neighborhood (which may necessitate making

$\mathcal{N}'(E^{\text{crit}})$ and ϵ_0 smaller as well). With this understood, the required extension of η_τ^ϵ from $\mathcal{N}(E^{\text{crit}})$ to a fiberwise Liouville form $\eta_{\tau,z}^\epsilon$ on $E|_{\mathcal{U}_z}$ exists for some neighborhood $\mathcal{U}_z \subset \Sigma \setminus \overline{\mathcal{U}'}$ of z .

Step 3 is then to choose a finite subcover $\{\mathcal{U}_z\}_{z \in I}$ of $\Sigma \setminus \mathcal{U}'$ with a subordinate partition of unity $\{\rho_z : \mathcal{U}_z \rightarrow [0, 1]\}_{z \in I}$ and define the desired extension by $\eta_\tau^\epsilon = \sum_{z \in I} (\rho_z \circ \Pi) \eta_{\tau,z}^\epsilon$. \square

Since the smaller neighborhood $\mathcal{U}' \subset \mathcal{U}$ in the above lemma contains $\text{Crit}(\varphi) \cup \partial\Sigma$, we can now relabel \mathcal{U}' as \mathcal{U} without loss of generality, and let $\mathcal{U}' \subset \mathcal{U}$ denote a still smaller neighborhood of $\text{Crit}(\varphi) \cup \partial\Sigma$ with closure in \mathcal{U} . Choose a smooth cutoff function

$$\beta : \Sigma \rightarrow [0, 1]$$

with compact support in \mathcal{U} such that $\beta|_{\mathcal{U}'} \equiv 1$, and define from this a family of smooth functions

$$F_\tau^\epsilon =: \varphi \circ \Pi + \epsilon(\beta \circ \Pi) f_\tau : E \rightarrow \mathbb{R}$$

for $\epsilon \in [0, \epsilon_0]$ and $\tau \in X$. Observe that $F_\tau^\epsilon = F = \varphi \circ \Pi$ outside of $E|_{\mathcal{U}}$. On $E \setminus \mathcal{N}'(E^{\text{crit}})$, we can also define the family of smooth 1-forms

$$\Theta_\tau^\epsilon = \Pi^* \sigma_\tau + \epsilon \eta_\tau^\epsilon,$$

which are Liouville for $\epsilon > 0$ sufficiently small (cf. Proposition 2.27).

Lemma 3.14. *For all $\epsilon > 0$ sufficiently small and all $\tau \in X$, F_τ^ϵ is J_τ -convex on $E|_{\mathcal{U}'}$, and*

$$dF_\tau^\epsilon \wedge \Theta_\tau^\epsilon \wedge d\Theta_\tau^\epsilon > 0 \quad \text{on } E \setminus (\mathcal{N}'(E^{\text{crit}}) \cup E|_{\text{Crit}(\varphi)}).$$

Proof. Consider first the region $E|_{\mathcal{U}'}$. Here Π is J_τ -holomorphic and $\beta \circ \Pi \equiv 1$, thus

$$-dF_\tau^\epsilon \circ J_\tau = \Pi^*(-d\varphi \circ j_\tau) + \epsilon(-df_\tau \circ J_\tau) = \Pi^* \sigma_\tau + \epsilon \lambda_\tau = \Theta_\tau^\epsilon.$$

Proposition 2.6 then implies that F_τ^ϵ is J_τ -convex on $E|_{\mathcal{U}'}$ for $\epsilon > 0$ sufficiently small, and consequently that Θ_τ^ϵ is contact on all the regular level sets of F_τ^ϵ in this region.

On $E \setminus (\mathcal{N}'(E^{\text{crit}}) \cup E|_{\mathcal{U}'})$, we compute

$$\begin{aligned} dF_\tau^\epsilon \wedge \Theta_\tau^\epsilon \wedge d\Theta_\tau^\epsilon &= (\Pi^* d\varphi + \epsilon d[(\beta \circ \Pi) f]) \wedge (\Pi^* \sigma_\tau + \epsilon \eta_\tau^\epsilon) \wedge (\Pi^* d\sigma_\tau + \epsilon d\eta_\tau^\epsilon) \\ &= \epsilon \Pi^*(d\varphi \wedge \sigma_\tau) \wedge d\eta_\tau^\epsilon + \mathcal{O}(\epsilon^2). \end{aligned}$$

This is positive for all $\epsilon > 0$ sufficiently small since outside of any neighborhood of $\text{Crit}(\varphi)$ and of E^{crit} respectively, $d\varphi \wedge \sigma_\tau$ and $d\eta_\tau^\epsilon|_{VE}$ can each be assumed to satisfy uniform positive lower bounds; note that the latter depends on the fact that $d\eta_\tau^\epsilon|_{VE} > 0$ holds even for $\epsilon = 0$ (cf. Lemma 3.11). \square

In light of Lemma 3.11, the family of 2-plane distributions ξ_τ^ϵ can now be extended from $\mathcal{N}(E^{\text{crit}})$ over the entirety of $E \setminus (E^{\text{crit}} \cup E|_{\text{Crit}(\varphi)})$ by setting

$$\xi_\tau^\epsilon = \ker dF_\tau^\epsilon \cap \ker \Theta_\tau^\epsilon.$$

Lemma 3.14 then implies that for all $\epsilon > 0$ sufficiently small, ξ_τ^ϵ defines a family of contact structures on the level sets of F_τ^ϵ . Observe that by construction, ξ_τ^ϵ is also preserved by J_τ on the neighborhood $E|_{\mathcal{U}'}$ of $E|_{\text{Crit}(\varphi)} \cup \partial_v E$, and it is preserved by J_τ^ϵ in $\mathcal{N}(E^{\text{crit}})$.

Lemma 3.15. *After possibly shrinking $\epsilon_0 > 0$, the family J_τ^ϵ defined near E^{crit} in §3.2 for $\epsilon \in [0, \epsilon_0]$ and $\tau \in X$ can be extended to a family of global almost complex structures on E*

that depend smoothly on ϵ , preserve ξ_τ^ϵ , and satisfy $J_\tau^\epsilon = J_\tau$ in some fixed neighborhood of $E|_{\text{Crit}(\varphi)} \cup \partial_v E$ for all ϵ and $J_\tau^0 \equiv J_\tau$. Moreover,

$$(3.6) \quad -dF_\tau^\epsilon \circ J_\tau^\epsilon = G_\tau^\epsilon \Theta_\tau^\epsilon \quad \text{along} \quad \partial_h E,$$

for a (uniquely determined) family of functions $G_\tau^\epsilon : \partial_h E \rightarrow (0, \infty)$ which depend smoothly on ϵ and satisfy $G_\tau^0 \equiv 1$.

Proof. Pick an open set $E^{\text{reg}} \subset E$ with closure disjoint from $E^{\text{crit}} \cup E|_{\text{Crit}(\varphi)} \cup \partial_v E$ such that

$$E = E^{\text{reg}} \cup E|_{\mathcal{U}'} \cup \mathcal{N}(E^{\text{crit}}).$$

Choose also a family g_τ of J_τ -invariant Riemannian metrics on E and let $H_\tau E^{\text{reg}} \subset TE^{\text{reg}}$ denote the g_τ -orthogonal complement of $VE|_{E^{\text{reg}}}$. By construction, $H_\tau E^{\text{reg}}$ is J_τ -invariant, and since $\xi_\tau^0 = VE$, we are free to assume $H_\tau E^{\text{reg}} \pitchfork \xi_\tau^\epsilon$ whenever $\epsilon \geq 0$ is sufficiently small. Then the projections $TE^{\text{reg}} \rightarrow VE$ along $H_\tau E$ restrict to a family of bundle isomorphisms $\Psi_\tau : \xi_\tau^\epsilon \rightarrow VE$, and there is a unique family of almost complex structures \hat{J}_τ^ϵ on E^{reg} defined by the conditions

$$\hat{J}_\tau^\epsilon|_{\xi_\tau^\epsilon} = \Psi_\tau^* J_\tau|_{VE}, \quad \hat{J}_\tau^\epsilon|_{H_\tau E} = J_\tau|_{H_\tau E}.$$

These preserve ξ_τ^ϵ and match J_τ for $\epsilon = 0$.

We next splice \hat{J}_τ^ϵ together with the existing families J_τ^ϵ on $\mathcal{N}(E^{\text{crit}})$ and $J_\tau^\epsilon := J_\tau$ on $E|_{\mathcal{U}'}$. For any point $p \in E$ with a complex structure J on $T_p E$ and sufficiently small J -antilinear map $Y : T_p E \rightarrow T_p E$, define

$$\Phi_J(Y) = \left(\mathbb{1} + \frac{1}{2} JY \right) J \left(\mathbb{1} + \frac{1}{2} JY \right)^{-1}.$$

This identifies a neighborhood of 0 in the space of J -antilinear maps on $T_p E$ with a neighborhood of J in the manifold of complex structures on $T_p E$, and moreover, if J preserves some subspace $V \subset T_p E$, then $\Phi_J(Y)$ also preserves V if and only if Y preserves V . On $E^{\text{reg}} \cap (\mathcal{N}(E^{\text{crit}}) \cup E|_{\mathcal{U}'})$, we may assume for sufficiently small $\epsilon \geq 0$ that \hat{J}_τ^ϵ and J_τ^ϵ are each C^0 -close to J_τ and therefore also to each other, so there exists a family of ξ_τ^ϵ -preserving J_τ^ϵ -antilinear bundle endomorphisms Y_τ^ϵ such that

$$\hat{J}_\tau^\epsilon = \Phi_{J_\tau^\epsilon}(Y_\tau^\epsilon),$$

and $Y_\tau^0 \equiv 0$. Now for any choice of smooth function $\psi : E \rightarrow [0, 1]$ that equals 1 outside $E|_{\mathcal{U}'} \cup \mathcal{N}(E^{\text{crit}})$ and has compact support in E^{reg} , a family of almost complex structures satisfying most of the desired properties can be defined by

$$(3.7) \quad J_\tau^\epsilon := \begin{cases} \hat{J}_\tau^\epsilon & \text{on } E^{\text{reg}} \setminus (\mathcal{N}(E^{\text{crit}}) \cup E|_{\mathcal{U}'}), \\ J_\tau^\epsilon & \text{on } (\mathcal{N}(E^{\text{crit}}) \cup E|_{\mathcal{U}'}) \setminus E^{\text{reg}}, \\ \Phi_{J_\tau^\epsilon}(\psi Y_\tau^\epsilon) & \text{on } E^{\text{reg}} \cap (\mathcal{N}(E^{\text{crit}}) \cup E|_{\mathcal{U}'}). \end{cases}$$

Notice that on the region where Π is holomorphic, $\beta = 1$ and $J_\tau^\epsilon = J_\tau$, we have

$$-dF_\tau^\epsilon \circ J_\tau^\epsilon = -d(\varphi \circ \Pi + \epsilon f_\tau) \circ J_\tau = \Pi^* \sigma_\tau + \epsilon \lambda_\tau = \Theta_\tau^\epsilon.$$

This applies in particular on a neighborhood of $E|_{\text{Crit}(\varphi)} \cup \partial_v E$, so that (3.6) is already established with $G_\tau^\epsilon = 1$ near $E|_{\text{Crit}(\varphi)}$. In order to achieve (3.6) everywhere else, we can modify the definition of J_τ^ϵ near $\partial_h E$ on a subbundle transverse to ξ_τ^ϵ . Indeed, observe first that on $\partial_h E$, the relation $T\Pi \circ J_\tau = j_\tau \circ T\Pi$ implies $-dF_\tau^0 \circ J_\tau^0 = \Pi^* \sigma_\tau$, and the latter

is nowhere zero away from $E|_{\text{Crit}(\varphi)}$, hence so are both $-dF_\tau^\epsilon \circ J_\tau^\epsilon|_{T(\partial_h E)}$ and Θ_τ^ϵ for $\epsilon \geq 0$ sufficiently small. Our goal will thus be to achieve

$$\ker(-dF_\tau^\epsilon \circ J_\tau^\epsilon) = \ker \Theta_\tau^\epsilon \quad \text{along} \quad \partial_h E.$$

Since Θ_τ^ϵ and $-dF_\tau^\epsilon \circ J_\tau^\epsilon$ both annihilate ξ_τ^ϵ , it suffices to find a 1-dimensional subbundle of $TE|_{\partial_h E}$ that intersects ξ_τ^ϵ trivially and is also annihilated by both. For Θ_τ^ϵ there is a clear choice: this 1-form is Liouville for sufficiently small $\epsilon > 0$, so its dual Liouville vector field V_τ^ϵ satisfies

$$\Theta_\tau^\epsilon(V_\tau^\epsilon) = d\Theta_\tau^\epsilon(V_\tau^\epsilon, V_\tau^\epsilon) = 0,$$

and we will see presently that it is not contained in VE , and therefore also not in ξ_τ^ϵ for $\epsilon > 0$ small, outside a neighborhood of $E|_{\text{Crit}(\varphi)}$. Indeed, working in a neighborhood of $\partial_h E$ where Π has no critical points and $\eta_\tau^\epsilon = \lambda_\tau$, let $H_\tau E \subset TE$ denote the $d\lambda_\tau$ -symplectic complement of VE . With respect to this splitting, write $V_\tau^\epsilon = v_\tau^\epsilon + h_\tau^\epsilon$ for $v_\tau^\epsilon \in VE$ and $h_\tau^\epsilon \in H_\tau E$. Writing $\Theta_\tau^\epsilon = \Pi^* \sigma_\tau + \epsilon \lambda_\tau$ and restricting the relation $\Theta_\tau^\epsilon = d\Theta_\tau^\epsilon(V_\tau^\epsilon, \cdot)$ to the subbundles VE and $H_\tau E$ then gives

$$\lambda_\tau|_{VE} = d\lambda_\tau(v_\tau^\epsilon, \cdot)|_{VE} \quad \text{and} \quad (\Pi^* \sigma_\tau + \epsilon \lambda_\tau)|_{H_\tau E} = (\Pi^* d\sigma_\tau + \epsilon d\lambda_\tau)(h_\tau^\epsilon, \cdot)|_{H_\tau E}.$$

The first relation identifies v_τ^ϵ as the ‘‘vertical Liouville vector field’’ V_{λ_τ} , defined on the smooth part of each fiber E_z as the Liouville vector field dual to $\lambda_\tau|_{TE_z}$. In particular, the vertical term does not depend on ϵ . The horizontal term h_τ^ϵ also has a well-behaved limit as $\epsilon \rightarrow 0$, determined by

$$\Pi^* \sigma_\tau|_{H_\tau E} = \Pi^* d\sigma_\tau(h_\tau^0, \cdot)|_{H_\tau E},$$

which means h_τ^0 is the horizontal lift $V_{\sigma_\tau}^\#$ of the Liouville vector field V_{σ_τ} on Σ , defined by $d\sigma_\tau(V_{\sigma_\tau}, \cdot) = \sigma_\tau$. The latter is nowhere zero away from $\text{Crit}(\varphi)$, implying that

$$V_\tau^0 := \lim_{\epsilon \rightarrow 0} V_\tau^\epsilon = V_{\lambda_\tau} + V_{\sigma_\tau}^\#$$

always has a nontrivial horizontal part on the region of interest. This establishes the claim that $V_\tau^\epsilon \notin \xi_\tau^\epsilon$ on this region for all $\epsilon \geq 0$ sufficiently small.

Now observe that at $\partial_h E$ for $\epsilon = 0$,

$$-dF_\tau^0(J_\tau V_\tau^0) = -d\varphi(\Pi_* J_\tau V_{\lambda_\tau} + \Pi_* J_\tau V_{\sigma_\tau}^\#) = -d\varphi(j_\tau \Pi_* V_{\sigma_\tau}^\#) = \sigma_\tau(V_{\sigma_\tau}) = d\sigma_\tau(V_{\sigma_\tau}, V_{\sigma_\tau}) = 0,$$

where we’ve again used the assumption that $T\Pi \circ J_\tau = j_\tau \circ T\Pi$ along $\partial_h E$. In other words, on $\partial_h E$ away from $E|_{\text{Crit}(\varphi)}$, V_τ^0 and $J_\tau V_\tau^0$ span a J_τ -complex subbundle of TE that is transverse to VE and intersects $\ker dF_\tau^0$ transversely in the subspace spanned by $J_\tau V_\tau^0$. At $E|_{\text{Crit}(\varphi)}$, the transversality fails because $V_{\sigma_\tau}^\#$ vanishes, but since $-dF_\tau^\epsilon \circ J_\tau^\epsilon = \Theta_\tau^\epsilon$ along $\partial_h E$ in this region, we also have

$$-dF_\tau^\epsilon(J_\tau^\epsilon V_\tau^\epsilon) = \Theta_\tau^\epsilon(V_\tau^\epsilon) = 0$$

here. It is therefore possible to modify the family J_τ^ϵ near $\partial_h E$ without changing it near $E|_{\text{Crit}(\varphi)}$ or changing its action on ξ_τ^ϵ anywhere so that it satisfies

$$J_\tau^\epsilon V_\tau^\epsilon \in \ker dF_\tau^\epsilon$$

everywhere along $\partial_h E$ for $\epsilon \geq 0$ sufficiently small. This identifies the kernels of $-dF_\tau^\epsilon \circ J_\tau^\epsilon$ and Θ_τ^ϵ along $\partial_h E$ and thus establishes (3.6) for a uniquely determined family of functions $G_\tau^\epsilon : \partial_h E \rightarrow (0, \infty)$ which necessarily equal 1 near $E|_{\text{Crit}(\varphi)}$. Since both families of 1-forms match $\Pi^* \sigma_\tau$ for $\epsilon = 0$, we also have $G_\tau^0 \equiv 1$. The modified family J_τ^ϵ can now be spliced together with the previously constructed family away from $\partial_h E$ using the same trick as in (3.7). \square

Lemma 3.16. *After replacing $\varphi : \Sigma \rightarrow \mathbb{R}$ by a function of the form $h \circ \varphi$ with $h' > 0$ and $h'' \gg 0$, the pairs $(J_\tau^\epsilon, F_\tau^\epsilon)$ become almost Stein structures for all $\tau \in X$ and all $\epsilon > 0$ sufficiently small.*

Proof. The functions F_τ^ϵ have critical points at E^{crit} and in $E|_{\text{Crit}(\varphi)}$, but are J_τ^ϵ -convex near both due to Lemmas 3.8 and 3.14. Outside these neighborhoods, the maximal J_τ^ϵ -complex subbundles on the level sets of F_τ^ϵ are the contact structures ξ_τ^ϵ , so F_τ^ϵ becomes J_τ^ϵ -convex after postcomposition with a sufficiently convex function, using Lemma 2.21.

It remains to check that $-dF_\tau^\epsilon \circ J_\tau^\epsilon$ restricts to contact forms on both $\partial_v E$ and $\partial_h E$. The former lies in the region where $-dF_\tau^\epsilon \circ J_\tau^\epsilon = \Theta_\tau^\epsilon = \Pi^* \sigma_\tau + \epsilon \lambda_\tau$, and Proposition 2.1 proves that the latter is contact on $\partial_v E$ for sufficiently small $\epsilon > 0$ since $\sigma_\tau|_{T(\partial\Sigma)} > 0$ and λ_τ is fiberwise Liouville. Using Proposition 2.2 similarly, Θ_τ^ϵ is also contact on $\partial_h E$ for small $\epsilon > 0$, so the contact condition on $\partial_h E$ follows from (3.6). \square

3.4. Interpolation of almost Stein structures. To complete the proof of Proposition 3.4, we need to relate the family of almost Stein structures $\{(J_\tau^\epsilon, F_\tau^\epsilon)\}_{\tau \in X, \epsilon \in (0, \epsilon_0]}$ constructed above to the given family $\{(J_\tau, f_\tau)\}_{\tau \in A}$. The functions f_τ were extended to all $\tau \in X$ in Lemma 3.12 but are only *fiberwise* J_τ -convex in general for $\tau \notin A$; on the other hand, all conditions that distinguish J_τ -convexity from its fiberwise counterpart are open, thus we can assume (J_τ, f_τ) are almost Stein structures for all τ in some open neighborhood $A' \subset X$ of A . The same can also be assumed for $(J_\tau^\epsilon, f_\tau)$ for any $\epsilon \in [0, \epsilon_0]$ if $\epsilon_0 > 0$ is sufficiently small. Now choose a cutoff function $\rho : X \rightarrow [0, 1]$ with support in A' and $\rho|_A \equiv 1$, and consider the family of interpolated functions

$$f_\tau^\epsilon := \rho(\tau) f_\tau + [1 - \rho(\tau)] F_\tau^\epsilon$$

for $\tau \in X$ and $\epsilon \in [0, \epsilon_0]$. These functions are J_τ^ϵ -convex everywhere when $\epsilon > 0$, but we still need to check that the remaining conditions of an almost Stein structure are satisfied for $\tau \in A' \setminus A$, i.e. that the interpolated Liouville forms

$$-df_\tau^\epsilon \circ J_\tau^\epsilon = \rho(\tau) (-df_\tau \circ J_\tau^\epsilon) + [1 - \rho(\tau)] (-dF_\tau^\epsilon \circ J_\tau^\epsilon)$$

are contact on both faces of ∂E . After shrinking $\epsilon_0 > 0$ further if necessary, this will follow from the next two lemmas.

Lemma 3.17. *For all $\tau \in A'$, $\epsilon > 0$ sufficiently small and $\rho \in [0, 1]$, the 1-forms $\rho(-df_\tau \circ J_\tau^\epsilon) + (1 - \rho)(-dF_\tau^\epsilon \circ J_\tau^\epsilon)$ restrict to contact forms on $\partial_v E$.*

Proof. In a neighborhood of $\partial_v E$, we have $J_\tau^\epsilon = J_\tau$ and thus $-df_\tau \circ J_\tau^\epsilon = \lambda_\tau$, and similarly, $F_\tau^\epsilon = \varphi \circ \Pi + \epsilon f_\tau$ and $T\Pi \circ J_\tau = j_\tau \circ T\Pi$ imply $-dF_\tau^\epsilon \circ J_\tau^\epsilon = \Pi^* \sigma_\tau + \epsilon \lambda_\tau$. The 1-form in question is thus

$$\rho \lambda_\tau + (1 - \rho)(\Pi^* \sigma_\tau + \epsilon \lambda_\tau) = [\rho + \epsilon(1 - \rho)] \left(\lambda_\tau + \frac{1 - \rho}{\rho + \epsilon(1 - \rho)} \Pi^* \sigma_\tau \right),$$

assuming $\epsilon > 0$ so that $\rho + \epsilon(1 - \rho) > 0$ for all ρ . Since λ_τ is fiberwise Liouville and defines a contact form on $\partial_v E$ and $\sigma_\tau|_{T(\partial\Sigma)} > 0$, the expression in parentheses is contact for all $\rho \in [0, 1]$ by Proposition 2.1. \square

Lemma 3.18. *The statement of Lemma 3.17 also holds for the restriction to $\partial_h E$.*

Proof. Near $\partial_h E$, Lemma 3.15 gives $-dF_\tau^\epsilon \circ J_\tau^\epsilon = G_\tau^\epsilon \Theta_\tau^\epsilon$ for a family of functions $G_\tau^\epsilon : \partial_h E \rightarrow (0, \infty)$ satisfying $G_\tau^0 \equiv 1$, while the 1-form $\Theta_\tau^\epsilon = \Pi^* \sigma_\tau + \epsilon \lambda_\tau$ is contact for $\epsilon > 0$ sufficiently

small due to Prop. 2.2. For $\epsilon = 0$, the interpolated 1-forms in question are thus

$$\rho\lambda_\tau + (1 - \rho)\Pi^*\sigma_\tau = \rho \left(\lambda_\tau + \frac{1 - \rho}{\rho} \Pi^*\sigma_\tau \right)$$

along $\partial_h E$, and these are contact for all $\rho > 0$ by another application of Prop. 2.2 since σ_τ is Liouville and $\lambda_\tau|_{T(\partial_h E)}$ satisfies the conditions of a Giroux form. The contact condition ceases to hold for $\rho = \epsilon = 0$, but since the condition is open, the lemma will follow from the claim that $\rho(-df_\tau \circ J_\tau^\epsilon) + (1 - \rho)(-dF_\tau^\epsilon \circ J_\tau^\epsilon)$ restricted to $\partial_h E$ is contact for every (ρ, ϵ) in a neighborhood of $(0, 0)$ excluding $(0, 0)$ itself. To see this, note first that since $\frac{1}{G_\tau^\epsilon}(-df_\tau \circ J_\tau^0) = \lambda_\tau$ and everything depends smoothly on ϵ , we can write

$$\frac{1}{G_\tau^\epsilon}(-df_\tau \circ J_\tau^\epsilon) = \lambda_\tau + \epsilon\gamma_\tau^\epsilon$$

for some family of smooth 1-forms $\{\gamma_\tau^\epsilon\}_{\tau \in X, \epsilon \in [0, \epsilon_0]}$. Our family of interpolated 1-forms on $\partial_h E$ can then be rewritten whenever $(\rho, \epsilon) \neq (0, 0)$ as

$$\begin{aligned} \rho(-df_\tau \circ J_\tau^\epsilon) + (1 - \rho)(-dF_\tau^\epsilon \circ J_\tau^\epsilon) &= \rho G_\tau^\epsilon (\lambda_\tau + \epsilon\gamma_\tau^\epsilon) + (1 - \rho)G_\tau^\epsilon (\Pi^*\sigma_\tau + \epsilon\lambda_\tau) \\ &= G_\tau^\epsilon (c\Pi^*\sigma_\tau + a\lambda_\tau + \epsilon\rho\gamma_\tau^\epsilon) =: G_\tau^\epsilon \mu, \end{aligned}$$

where we are abbreviating $c = c(\rho) := 1 - \rho$ and $a = a(\rho, \epsilon) := \rho + \epsilon(1 - \rho)$. Notice that while $c(\rho)$ approaches 1, $a(\rho, \epsilon)$ and $\epsilon\rho/a(\rho, \epsilon)$ each decay to 0 as $(\rho, \epsilon) \rightarrow (0, 0)$. Since $\Pi^*\sigma_\tau \wedge d\lambda_\tau = 0$ and $\lambda_\tau \wedge \Pi^*d\sigma_\tau > 0$ by the fiberwise Giroux condition on λ_τ , we then find that

$$\begin{aligned} \mu \wedge d\mu &= (c\Pi^*\sigma_\tau + a\lambda_\tau + \epsilon\rho\gamma_\tau^\epsilon) \wedge (c\Pi^*d\sigma_\tau + a d\lambda_\tau + \epsilon\rho d\gamma_\tau^\epsilon) \\ &= ac \left[\lambda_\tau \wedge \Pi^*d\sigma_\tau + \frac{a}{c}\lambda_\tau \wedge d\lambda_\tau + \frac{\epsilon\rho}{c} \left(\lambda_\tau \wedge d\gamma_\tau^\epsilon + \gamma_\tau^\epsilon \wedge d\lambda_\tau + \frac{\epsilon\rho}{a}\gamma_\tau^\epsilon \wedge d\gamma_\tau^\epsilon \right) \right] \end{aligned}$$

is positive as soon as (ρ, ϵ) gets close enough to $(0, 0)$. \square

With this, the pairs $(J_\tau^\epsilon, f_\tau^\epsilon)$ for all $\tau \in X$ and $\epsilon > 0$ sufficiently small are seen to be almost Stein structures that match (J_τ, f_τ) for $\tau \in A$, so the proof of Proposition 3.4 (and therefore also of Theorem 3.1) is now complete.

4. A SYMPLECTIC MODEL OF A COLLAR NEIGHBORHOOD WITH CORNERS

Throughout this section, assume (M', ξ) is a closed connected contact 3-manifold, and $M \subset M'$ is a compact connected 3-dimensional submanifold $M \subset M'$, possibly with boundary, on which ξ is supported by a spinal open book

$$\pi := \left(\pi_\Sigma : M_\Sigma \rightarrow \Sigma, \pi_P : M_P \rightarrow S^1, \{m_T\}_{T \subset \partial M} \right).$$

The immediate purpose of this section is to construct a precise symplectic model of a collar neighborhood of the form $(-\epsilon, 0] \times M'$ in the symplectization of (M', ξ) , designed such that spine removal cobordisms can be defined via an easy modification of the model. The intuition for the construction comes from the neighborhood of ∂E when $\Pi : E \rightarrow \Sigma$ is a bordered Lefschetz fibration that fills a spinal open book—however, it will not be necessary to assume in the following that (M', ξ) is symplectically fillable, as we will instead make use of the trivial observation that *every* closed contact manifold arises as the convex boundary of a noncompact subset of its own symplectization. The model we construct will thus be a noncompact 4-manifold E' whose boundary has two smooth faces

$$\partial E' = \partial_v E' \cup \partial_h E',$$

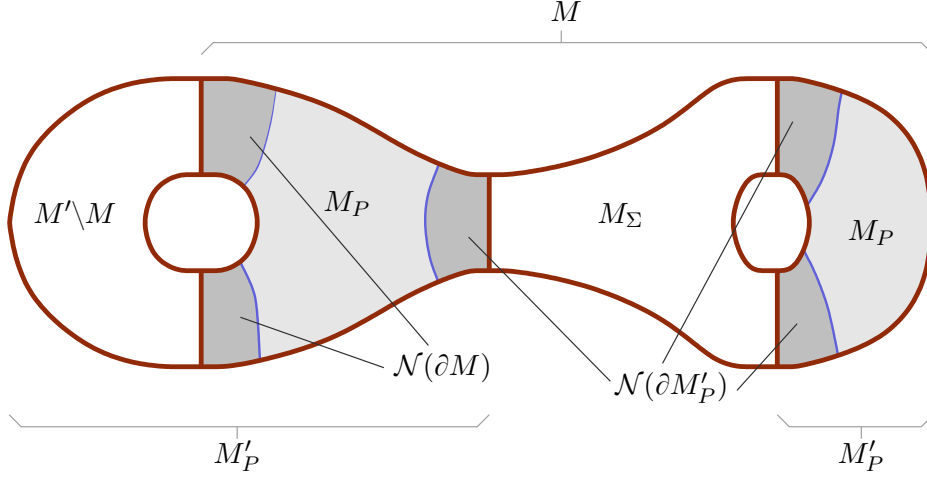


FIGURE 4. A schematic picture of the closed manifold M' with compact subdomains $M = M_P \cup M_\Sigma \subset M'$ and $M'_P \subset M'$, together with the various collar neighborhoods $\mathcal{N}(\partial M)$ and $\mathcal{N}(\partial M'_P)$ of ∂M_P .

interpreted as the vertical and horizontal boundaries respectively of a (locally defined) symplectic fibration, such that the smoothed contact boundary of E' can be identified with (M', ξ) . We will elaborate further on this model in [LVW] by attaching cylindrical ends to both its fibers and its base, producing the so-called *double completion* of E' , which will admit an abundance of holomorphic curves modeled after the pages of π . These curves generate the moduli space needed for classifying fillings as in Theorems A and B.

4.1. The Liouville collar.

4.1.1. *The collar and its boundary.* We shall denote the union of the paper with the “rest” of M' by

$$M'_P := M_P \cup (M' \setminus M) = M' \setminus \overset{\circ}{M}_\Sigma \subset M',$$

hence M' is the union of M_Σ with M'_P along their common boundary $\partial M_\Sigma = \partial M'_P$, a disjoint union of 2-tori. Recall from §2.2 the collar neighborhoods $\mathcal{N}(\partial\Sigma)$, $\mathcal{N}(\partial M_\Sigma)$ and $\mathcal{N}(\partial M_P)$ with their coordinate systems (s, ϕ) , (s, ϕ, θ) and (ϕ, t, θ) respectively. We will denote by

$$\mathcal{N}(\partial M) \subset M_P$$

the neighborhood of ∂M in M defined as the union of all components of $\mathcal{N}(\partial M_P)$ that touch ∂M . Similarly, the union of components of $\mathcal{N}(\partial M_P)$ that are disjoint from ∂M will be denoted by

$$\mathcal{N}(\partial M'_P) \subset M'_P,$$

as this forms a collar neighborhood of $\partial M'_P$ in M'_P . For assistance in keeping track of this notation, see Figure 4.

Now since $\partial M_\Sigma = \partial M'_P$, we can use the collars $\mathcal{N}(\partial M_\Sigma) = (-1, 0] \times \partial M_\Sigma$ and $\mathcal{N}(\partial M'_P) = (-1, 0] \times \partial M'_P$ to define a diffeomorphism

$$\begin{aligned} \Phi : (-1, 0] \times \mathcal{N}(\partial M_\Sigma) &\rightarrow (-1, 0] \times \mathcal{N}(\partial M'_P) \\ (t, (s, x)) &\mapsto (s, (t, x)), \end{aligned}$$

and then use this as a gluing map to define (see Figure 5)

$$E' := ((-1, 0] \times M_\Sigma) \cup_\Phi ((-1, 0] \times M'_P),$$

along with the distinguished subdomain

$$E := ((-1, 0] \times M_\Sigma) \cup_\Phi ((-1, 0] \times M_P) \subset E'.$$

This construction makes E' and E into smooth noncompact 4-manifolds with boundary and codimension 2 corners. The boundary of E' consists of two smooth faces

$$\partial E' = \partial_v E' \cup \partial_h E'$$

defined as follows:

- The **vertical boundary** $\partial_v E'$ is $\{0\} \times M'_P$, so it is a copy of M'_P . We will denote the resulting collar neighborhood of the vertical boundary by

$$\mathcal{N}(\partial_v E') := (-1, 0] \times M'_P \subset E',$$

and denote the coordinate on the first factor by s . We will also want to consider the distinguished subset

$$\partial_v E := \{0\} \times M_P \subset \partial_v E'$$

and the corresponding collar

$$\mathcal{N}(\partial_v E) := (-1, 0] \times M_P \subset \mathcal{N}(\partial_v E'),$$

which are the same as $\partial_v E'$ and $\mathcal{N}(\partial_v E')$ respectively if $\partial M = \emptyset$.

- The **horizontal boundary** $\partial_h E'$ is $\{0\} \times M_\Sigma$, a copy of M_Σ , and it can also be denoted by $\partial_h E := \partial_h E'$ since it lies in the subdomain E . The resulting collar neighborhood of this face will be denoted by

$$\mathcal{N}(\partial_h E) := \mathcal{N}(\partial_h E') := (-1, 0] \times M_\Sigma \subset E,$$

with the coordinate on the first factor denoted by t .

Notice that $\partial_v E' \cup \partial_h E'$ is naturally homeomorphic to M' , and similarly $\partial_v E \cup \partial_h E$ is homeomorphic to M , in both cases by a homeomorphism that identifies the corner $\partial_v E' \cap \partial_h E' = \partial_v E \cap \partial_h E$ with $\partial M_\Sigma = \partial M'_P = M_\Sigma \cap M_P$. Each connected component of the neighborhood

$$\mathcal{N}(\partial_v E \cap \partial_h E) := \mathcal{N}(\partial_v E) \cap \mathcal{N}(\partial_h E) \subset E$$

of this corner carries coordinates

$$(s, \phi, t, \theta) \in (-1, 0] \times S^1 \times (-1, 0] \times S^1 \subset \mathcal{N}(\partial_v E \cap \partial_h E),$$

as the construction of the gluing map guarantees that each of these coordinates is unambiguously defined. We assign to E' and E the orientation determined by this coordinate system. A similar coordinate system exists on each connected component of

$$\mathcal{N}(\partial \partial_v E) := (-1, 0] \times \mathcal{N}(\partial M) \subset \mathcal{N}(\partial_v E) \subset E.$$

On the collars $\mathcal{N}(\partial_h E)$ and $\mathcal{N}(\partial_v E)$, one can separately define fibrations

$$\Pi_h : \mathcal{N}(\partial_h E) = (-1, 0] \times (\Sigma \times S^1) \rightarrow \Sigma : (t, (z, \theta)) \mapsto \pi_\Sigma(z, \theta) = z,$$

and

$$\Pi_v : \mathcal{N}(\partial_v E) = (-1, 0] \times M_P \rightarrow (-1, 0] \times S^1 : (s, x) \mapsto (s, \pi_P(x)).$$

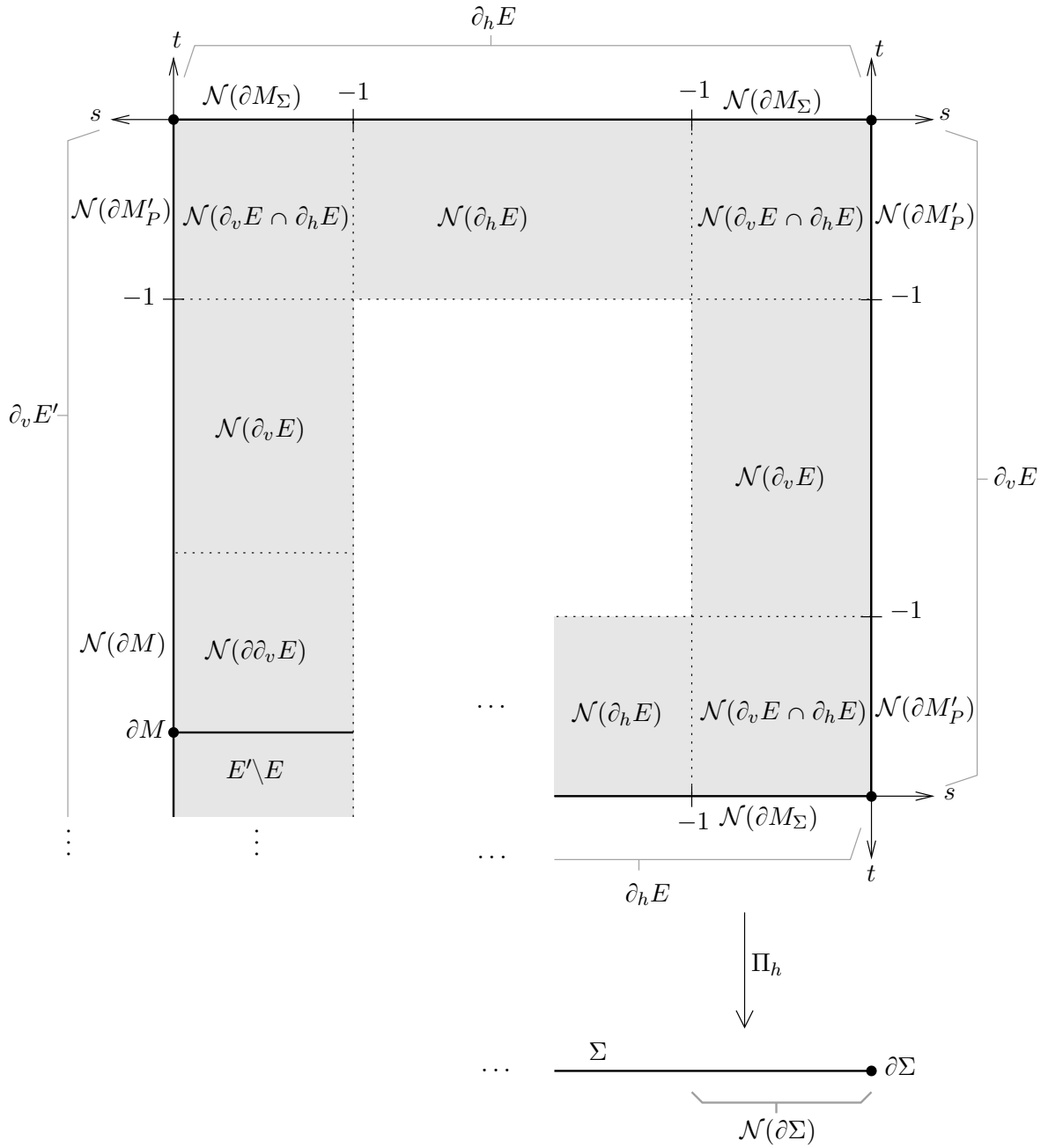


FIGURE 5. The domain E' with its boundary faces and collar neighborhoods, shown together with a portion of the fibration $\Pi_h : \mathcal{N}(\partial_h E) \rightarrow \Sigma$. In this example, M_P contains at least two connected components, one (shown at the right) that touches two separate spinal components but not the boundary, and another (at the left) that does touch ∂M .

On the region where the domains of these two fibrations overlap, we can write them in (s, ϕ, t, θ) -coordinates as

$$(4.1) \quad \Pi_h(s, \phi, t, \theta) = (s, \phi), \quad \Pi_v(s, \phi, t, \theta) = (s, m\phi).$$

While it may not be true in general that Π_h and Π_v can be fit together to define a global fibration on E , they have the same fibers on the region of overlap and thus give rise to a well-defined **vertical subbundle**

$$VE := \ker T\Pi_h \text{ or } \ker T\Pi_v \subset TE,$$

which on $\mathcal{N}(\partial_h E)$ is spanned by the vector fields ∂_t and ∂_θ . Figure 5 has been drawn so that the fibers can be represented as vertical lines in the picture.

4.1.2. *The Liouville structure on E' .* We will use the fibrations Π_v and Π_h to construct a Liouville structure on E via the Thurston trick as in §2.1, and then extend it to E' using the given contact structure on M' .

Fix a Liouville form σ on Σ that takes the form

$$\sigma = me^s d\phi \quad \text{on } \mathcal{N}(\partial\Sigma),$$

where $m \in \mathbb{N}$ is the *multiplicity* of $\pi_P : M_P \rightarrow S^1$ at its boundary component adjacent to the relevant component of $\mathcal{N}(\partial M_\Sigma)$; recall that this number may differ on distinct connected components of $\mathcal{N}(\partial\Sigma)$, cf. §2.2. We will also use σ to denote the pullback of this Liouville form under the trivial bundle projection $\Pi_h : \mathcal{N}(\partial_h E) \rightarrow \Sigma$, and since $\pi_P(\phi, t, \theta) = m\phi$ on $\mathcal{N}(\partial M_P)$, σ extends globally to a 1-form on E satisfying

$$\sigma = e^s d\pi_P \quad \text{on } \mathcal{N}(\partial_v E),$$

where we are abusing notation slightly by using $\pi_P : \mathcal{N}(\partial_v E) \rightarrow S^1$ to denote the composition of the fibration $\pi_P : M_P \rightarrow S^1$ with the obvious projection $\mathcal{N}(\partial_v E) = (-1, 0] \times M_P \rightarrow M_P$, hence defining $d\pi_P$ as a real-valued 1-form on $\mathcal{N}(\partial_v E)$.

We next define a 1-form on E that can be regarded as a *fiberwise Liouville* structure with respect to the fibrations Π_h and Π_v . By Lemma 2.7, there exists a 1-form λ on M_P such that $d\lambda$ is positive on all fibers of $\pi_P : M_P \rightarrow S^1$ and

$$\lambda = e^t d\theta \quad \text{on } \mathcal{N}(\partial M_P).$$

Using the same symbol to denote the pullback of λ via the projection $\mathcal{N}(\partial_v E) = (-1, 0] \times M_P \rightarrow M_P$, we can then extend λ to a global 1-form on E satisfying

$$\lambda = e^t d\theta \quad \text{on } \mathcal{N}(\partial_h E).$$

It is fiberwise Liouville in the sense that $d\lambda|_{VE} > 0$ everywhere on E , and since $\lambda|_{T(\partial_h E)} = d\theta$, the boundaries of the fibers of Π_h are positive with respect to λ and are annihilated by $d\lambda|_{T(\partial_h E)}$.

We can now apply the Thurston trick: for any constant $K \geq 0$, we define a 1-form λ_K by

$$\lambda_K := K\sigma + \lambda.$$

Corollary 2.5 in conjunction with Remark 4.1 below then provides a constant $K_0 > 0$ such that $d\lambda_K$ is symplectic everywhere on E for each $K \geq K_0$. Near the boundary, we have

$$(4.2) \quad \lambda_K = K\sigma + e^t d\theta \text{ on } \mathcal{N}(\partial_h E), \quad \text{and} \quad \lambda_K = Ke^s d\pi_P + \lambda \text{ on } \mathcal{N}(\partial_v E),$$

so in particular

$$(4.3) \quad \lambda_K = Kme^s d\phi + e^t d\theta \quad \text{on } \mathcal{N}(\partial_v E \cup \partial_h E) \cup \mathcal{N}(\partial\partial_v E).$$

Remark 4.1. The noncompactness of E does not pose any problem in the above use of the Thurston trick: the reason is that if we fix on E any Riemannian metric that is independent of the s - and/or t -coordinates wherever these are defined, then $|d\lambda \wedge d\lambda|$ is bounded above and $d\sigma \wedge d\lambda$ is bounded away from zero. This observation will be even more useful when we discuss the double completion in [LVW].

We will always assume from now on that $K \geq K_0$ so that $d\lambda_K$ is symplectic, and we will occasionally require further increases in the value of K_0 for convenience. There is now a Liouville vector field V_K on $(E, d\lambda_K)$ defined via the condition

$$d\lambda_K(V_K, \cdot) \equiv \lambda_K.$$

From (4.2) we compute

$$(4.4) \quad V_K = V_\sigma + \partial_t \quad \text{on } \mathcal{N}(\partial_h E),$$

where V_σ denotes the Liouville vector field on Σ dual to σ , and from (4.3),

$$V_K = \partial_s + \partial_t \quad \text{on } \mathcal{N}(\partial_v E \cap \partial_h E) \cup \mathcal{N}(\partial\partial_v E).$$

Lemma 4.2. *For all $K > 0$ sufficiently large, $ds(V_K) > 0$ on $\mathcal{N}(\partial_v E)$.*

Proof. It is equivalent to show that the restriction of $\lambda_K = Ke^s d\pi_P + \lambda$ to $\{s\} \times M_P$ for each $s \in (-1, 0]$ is a positive contact form. Since $e^s d\pi_P$ is the pullback via $\pi_P : M_P \rightarrow S^1$ of a volume form on S^1 for each fixed $s \in [-1, 0]$, the result follows from Proposition 2.1. \square

In light of the lemma, we shall assume from now on that $K_0 > 0$ is large enough to ensure $ds(V_K) > 0$ for all $K \geq K_0$. Before extending λ_K to the rest of E' , we must make a minor adjustment in the neighborhood of $(-1, 0] \times \partial M \subset \mathcal{N}(\partial_v E)$.

Lemma 4.3. *There exists a smooth homotopy of Liouville forms $\{\lambda_K^\tau\}_{\tau \in [0, 1]}$ on E with the following properties:*

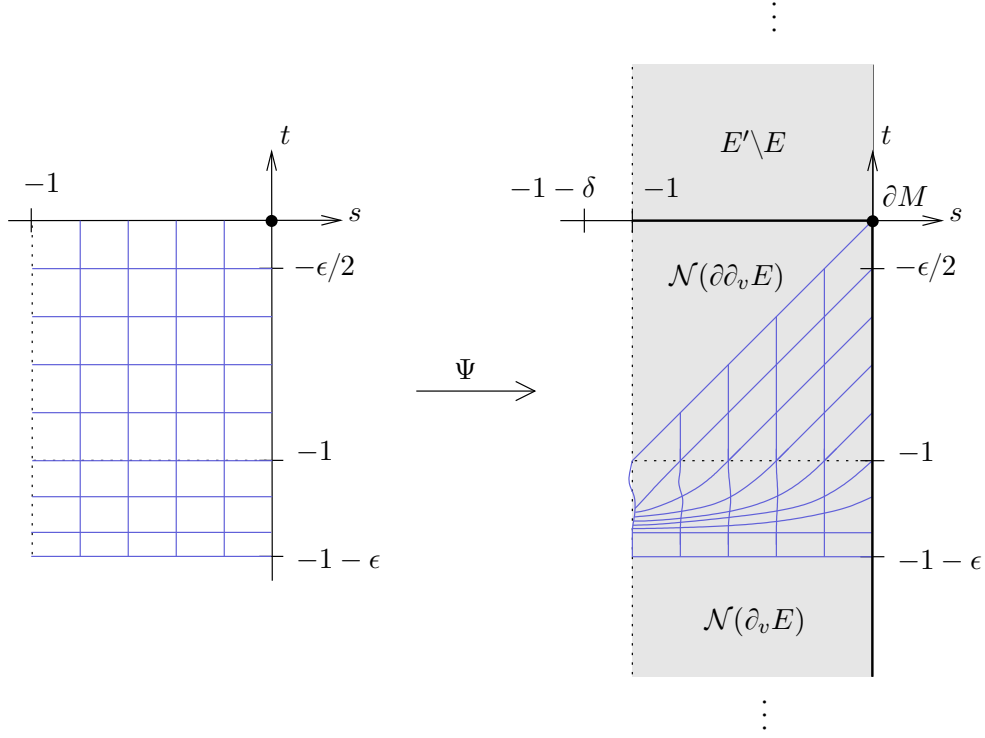
- (1) $\lambda_K^0 \equiv \lambda_K$;
- (2) The restrictions of λ_K^τ to $T(\partial_v E)$ are identical for every $\tau \in [0, 1]$;
- (3) $\lambda_K^\tau \equiv \lambda_K$ outside a small open neighborhood of $\mathcal{N}(\partial\partial_v E)$ for all $\tau \in [0, 1]$;
- (4) For each $\tau \in [0, 1]$, the Liouville vector field V_K^τ determined by λ_K^τ satisfies $ds(V_K^\tau) > 0$ on $\mathcal{N}(\partial_v E)$;
- (5) $\lambda_K^1 = e^s (Km d\phi + e^t d\theta)$ near $(-1, 0] \times \partial M \subset \mathcal{N}(\partial_v E)$, where $m \in \mathbb{N}$ is the multiplicity of π_P at the relevant component of ∂M .

Proof. Working in (ϕ, t, θ) -coordinates on a connected component of $\mathcal{N}(\partial M)$, let $\widehat{\mathcal{N}}(\partial M) \subset M$ denote a slightly expanded collar neighborhood in which the t -coordinate takes values in $(-1 - \epsilon, 0]$ for some $\epsilon > 0$ small. Let us similarly extend the s -coordinate to the interval $(-1 - \delta, 0]$ and consider the expanded domain

$$\widehat{\mathcal{N}}(\partial\partial_v E) := (-1 - \delta, 0] \times \widehat{\mathcal{N}}(\partial M)$$

for some $\delta > 0$ small enough so that $\lambda_K = Kme^s d\phi + \lambda$ is still a Liouville form on this domain and its Liouville vector field V_K is still transverse to all hypersurfaces of the form $\{s = \text{const}\}$. Notice that in the region $\{t \geq -1\} \subset \widehat{\mathcal{N}}(\partial\partial_v E)$, we have $\lambda_K = Kme^s d\phi + e^t d\theta$ and thus $V_K = \partial_s + \partial_t$. Now if $\epsilon > 0$ is sufficiently small, we can assume that the flow $\Phi_{V_K}^\rho$ of V_K in $\widehat{\mathcal{N}}(\partial\partial_v E)$ for times $\rho \in [-1, 0]$ is well defined on the small collar

$$\{t \geq -\epsilon/2\} \subset \mathcal{N}(\partial M) \subset \partial_v E.$$

FIGURE 6. The embedding Ψ in the proof of Lemma 4.3.

Choose a smooth vector field V on $\widehat{\mathcal{N}}(\partial \partial_v E)$ with the following properties:

- (1) $V \equiv V_K$ throughout $\mathcal{N}(\partial \partial_v E)$ and also in the region obtained by flowing $\{t \geq -\epsilon/2\} \subset \mathcal{N}(\partial M)$ backwards from time 0 to time -1 ;
- (2) $ds(V) > 0$ is close to 1 everywhere;
- (3) $V \equiv \partial_s$ in a neighborhood of $\{t = -1 - \epsilon\}$.

Using the flow Φ_V^\bullet of V , define the embedding (see Figure 6)

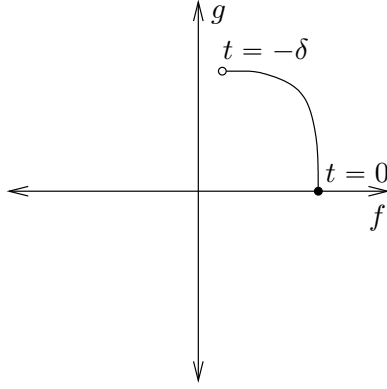
$$(-1, 0] \times S^1 \times (-1 - \epsilon, 0] \times S^1 \xrightarrow{\Psi} \widehat{\mathcal{N}}(\partial \partial_v E) : (s, \phi, t, \theta) \mapsto \Phi_V^s(\phi, t, \theta).$$

Identifying the domain of Ψ with the obvious collar neighborhood in $\mathcal{N}(\partial_v E)$, this map equals the identity near $\{t = -1 - \epsilon\}$ and at $\{s = 0\}$, and by deforming the vector field V we can also find a smooth isotopy of embeddings $\{\Psi_\tau\}_{\tau \in [0, 1]}$ with both of these properties such that $\Psi_1 = \Psi$ and $\Psi_0 = \text{Id}$. The desired family of Liouville forms can then be defined on this collar by

$$\lambda_K^\tau = \Psi_\tau^* \lambda_K$$

and extended to the rest of E as λ_K . In particular, we have $\lambda_K^1 = e^s (Km d\phi + e^t d\theta)$ for $t \geq -\epsilon/2$ since Ψ redefines the s -coordinate via the flow of the Liouville vector field. Since $V_K = \partial_s + \partial_t$ on $\mathcal{N}(\partial \partial_v E)$, the condition $ds(V_K^\tau) > 0$ is easily achieved as long as δ and ϵ are both sufficiently small. \square

Let us now replace λ_K with λ_K^1 from the lemma, so as to assume without loss of generality that $\lambda_K = e^s (Km d\phi + e^t d\theta)$ on $(-1, 0] \times S^1 \times (-\delta, 0] \times S^1 \subset \mathcal{N}(\partial \partial_v E)$ for some $\delta > 0$. We

FIGURE 7. The path $t \mapsto (f(t), g(t))$ for $-\delta < t \leq 0$.

then make one further modification on the same region and redefine λ_K in the form

$$\lambda_K = e^s [f(t) d\theta + Kmg(t) d\phi],$$

where $f, g : (-\delta, 0] \rightarrow [0, \infty)$ are smooth functions chosen such that (see Figure 7)

- $(f(t), g(t)) = (e^t, 1)$ for t near $-\delta$;
- $f'g - fg' > 0$;
- $f(0) = 1$ and $g(0) = 0$;
- $f'(0) = 0$.

These conditions guarantee that $\alpha' := f(t) d\theta + Kmg(t) d\phi$ defines a positive contact form on $S^1 \times (-\delta, 0] \times S^1 \subset \mathcal{N}(\partial M)$ satisfying $\alpha'(\partial_\phi) = 0$ and $d\alpha'(\partial_\theta, \cdot) = 0$ at ∂M . One can now extend α' smoothly beyond ∂M so that it defines a contact form for ξ on $M' \setminus M$. The corresponding extension of λ_K is defined by

$$\lambda_K := e^s \alpha' \quad \text{on } (-1, 0] \times (M' \setminus M) \subset \mathcal{N}(\partial_v E').$$

The corresponding Liouville vector field on $(-1, 0] \times (M' \setminus M)$ is simply ∂_s .

4.1.3. Contact hypersurfaces and smoothing corners. It is immediate from the above constructions that the Liouville vector field V_K is transverse to both $\partial_v E'$ and $\partial_h E'$, so smoothing the corners makes $\partial E'$ into a contact hypersurface. Moreover, the fiberwise Liouville condition on λ_K and the specific way that it was modified in $\mathcal{N}(\partial \partial_v E)$ mean that the induced contact structure on the smoothing of $\partial_h E \cup \partial_v E$ will be isotopic to one supported by π , hence the contact structure on $\partial E'$ is isotopic to ξ after identifying the latter with M' .

To define the smoothing more precisely, choose a pair of smooth functions $F, G : (-1, 1) \rightarrow (-1, 0]$ that satisfy the following conditions:

- $(F(\rho), G(\rho)) = (\rho, 0)$ for $\rho \leq -1/4$;
- $(F(\rho), G(\rho)) = (0, -\rho)$ for $\rho \geq 1/4$;
- $G'(\rho) < 0$ for $\rho > -1/4$;
- $F'(\rho) > 0$ for $\rho < 1/4$.

Now let

$$M^0 \subset E'$$

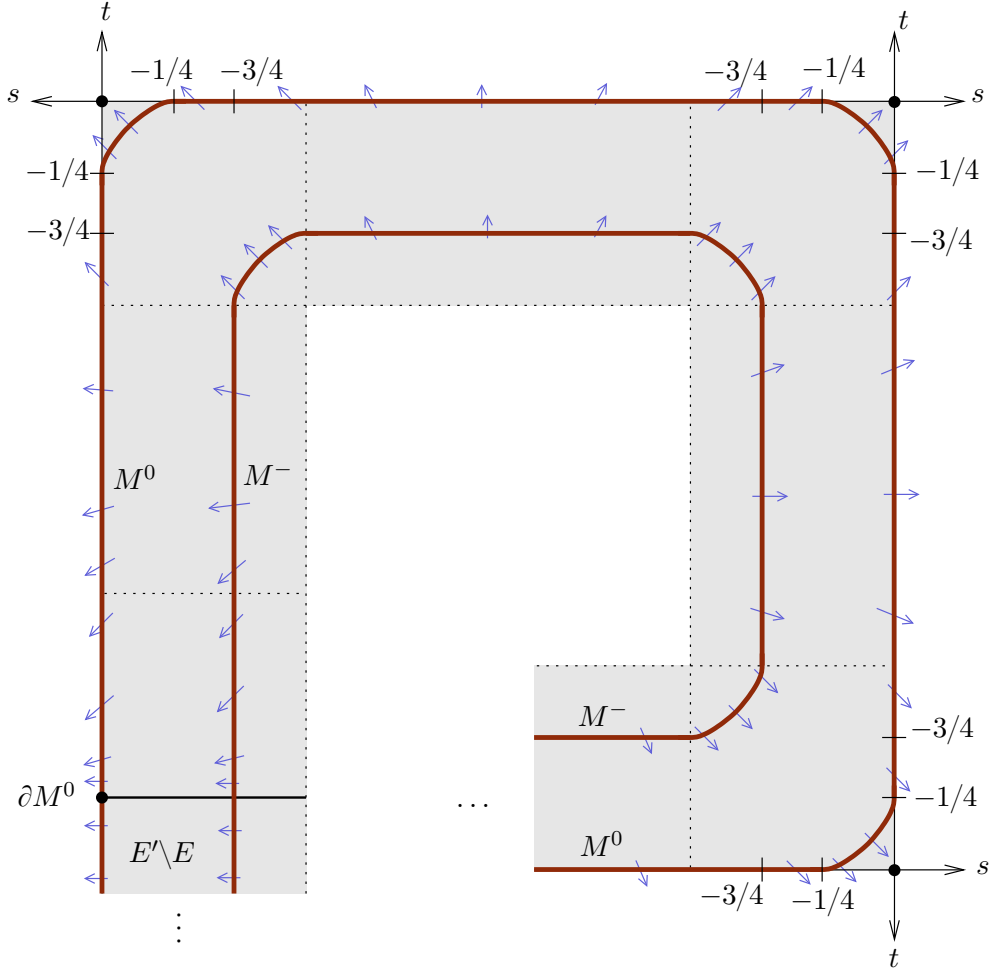


FIGURE 8. The smoothed hypersurfaces M^0 and M^- sitting inside the same model of E' as shown in Figure 5, together with the transverse Liouville vector field V_K .

denote the smooth hypersurface obtained from $\partial E'$ by replacing $\partial E' \cap \mathcal{N}(\partial_v E \cap \partial_h E)$ in (s, ϕ, t, θ) -coordinates with

$$(4.5) \quad \left\{ (F(\rho), \phi, G(\rho), \theta) \mid \phi, \theta \in S^1, -1 < \rho < 1 \right\};$$

see Figure 8. This smoothing is transverse to $V_K = \partial_s + \partial_t$ by construction, thus M^0 is a contact hypersurface and inherits the contact structure

$$\xi_0 := \ker \alpha_0, \quad \alpha_0 := \lambda_K|_{TM^0}.$$

By translating M^0 a distance of $-3/4$ in both the s - and t -coordinates, one obtains another contact hypersurface

$$(M^-, \xi_-) \subset (E', d\lambda_K)$$

which contains portions of the two hypersurfaces $\{-3/4\} \times M_\Sigma \subset \mathcal{N}(\partial_h E)$ and $\{-3/4\} \times M'_P \subset \mathcal{N}(\partial_v E')$ and a translated copy of (4.5) replacing the neighborhood of their intersection (see

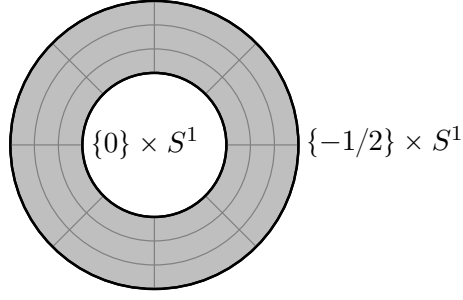


FIGURE 9. The diffeomorphism $\psi : [-1/2, 0] \times S^1 \rightarrow \mathbb{D}^2 \setminus \mathring{\mathbb{D}}_{1/2}^2$.

the inner hypersurface in Figure 8). Since (M^-, ξ_-) and (M^0, ξ_0) can evidently be connected by a smooth 1-parameter family of contact hypersurfaces in $(E', d\lambda_K)$, their contact structures are isotopic, so in particular ξ_- is isotopic to ξ after a suitable identification of M^- with M' .

4.2. Spine removal cobordisms. In this section we use the model $(E', d\lambda_K)$ with contact hypersurfaces (M^-, ξ_-) and (M^0, ξ_0) constructed in §4.1 to prove Theorem 1.25. In particular, we will enlarge E' in order to construct a symplectic spine removal cobordism whose negative weakly contact boundary is the contact hypersurface (M^-, ξ_-) .

Fix a decomposition of Σ into open and closed subsets

$$\Sigma = \Sigma^{\text{rem}} \amalg \Sigma^{\text{oth}},$$

and assume Σ^{rem} is nonempty. Fix also a trivialization $M_\Sigma = \Sigma \times S^1$, so in particular $\pi_\Sigma^{-1}(\Sigma^{\text{rem}}) = \Sigma^{\text{rem}} \times S^1$. The choice of decomposition $\Sigma = \Sigma^{\text{rem}} \amalg \Sigma^{\text{oth}}$ splits the horizontal boundary $\partial_h E$ into a disjoint union

$$\partial_h E = \partial_h^{\text{rem}} E \amalg \partial_h^{\text{oth}} E := (\Sigma^{\text{rem}} \times S^1) \amalg (\Sigma^{\text{oth}} \times S^1),$$

and the collar $\mathcal{N}(\partial_h E)$ decomposes accordingly as

$$\mathcal{N}(\partial_h E) = \mathcal{N}(\partial_h^{\text{rem}} E) \amalg \mathcal{N}(\partial_h^{\text{oth}} E).$$

Recall that $\lambda_K = K\sigma + e^t d\theta$ in $\mathcal{N}(\partial_h E)$.

We will now modify E' by attaching a generalized notion of a “symplectic handle” to $\partial_h^{\text{rem}} E$. Choose a diffeomorphism

$$\psi : [-1/2, 0] \times S^1 \xrightarrow{\cong} \mathbb{D}^2 \setminus \mathring{\mathbb{D}}_{1/2}^2,$$

where $\mathbb{D}_{1/2}^2$ denotes the closed disk of radius $1/2$ inside the unit disk $\mathbb{D}^2 \subset \mathbb{C}$, and assume ψ maps $\{-1/2\} \times S^1$ to $\partial \mathbb{D}^2$; see Figure 9. Using the obvious coordinates (t, θ) on $[-1/2, 0] \times S^1$, let $\omega_{\mathbb{D}}$ denote any area form on \mathbb{D}^2 that restricts to $\psi_*(e^t dt \wedge d\theta)$ outside of $\mathbb{D}_{1/2}^2$. We can then define a new symplectic manifold with boundary and corners by

$$(\tilde{E}', \tilde{\omega}_K) := (E', d\lambda_K) \cup_\Psi (\Sigma^{\text{rem}} \times \mathbb{D}^2, K d\sigma + \omega_{\mathbb{D}}),$$

where $d\sigma$ and $\omega_{\mathbb{D}}$ are each identified with their pullbacks to $\Sigma^{\text{rem}} \times \mathbb{D}^2$ via the obvious projections, and the gluing map is defined by

$$\Psi : \mathcal{N}(\partial_h^{\text{rem}} E) \supset (-1/2, 0] \times \Sigma^{\text{rem}} \times S^1 \hookrightarrow \Sigma^{\text{rem}} \times \mathbb{D}^2 : (t, z, \theta) \mapsto (z, \psi(t, \theta)).$$

Schematic pictures of this modification are shown in Figures 10 and 11, for cases where Σ^{rem} has one or two connected components respectively. Since $\mathcal{N}(\partial_h^{\text{rem}} E)$ lies entirely in the subdomain $E \subset E'$, we can define a corresponding subdomain

$$\tilde{E} \subset \tilde{E}'$$

by attaching $\Sigma^{\text{rem}} \times \mathbb{D}^2$ in this way to E instead of E' . The boundary of \tilde{E}' now has two smooth faces $\partial \tilde{E}' = \partial_h \tilde{E}' \cup \partial_v \tilde{E}'$, where the “horizontal” boundary is

$$\partial_h \tilde{E}' := \partial_h \tilde{E} := \partial_h^{\text{oth}} E,$$

and the “vertical” boundary

$$\partial_v \tilde{E}' = \partial_v E' \cup (\partial \Sigma^{\text{rem}} \times \mathbb{D}^2)$$

is obtained from $\partial_v E'$ by gluing in $\partial \Sigma^{\text{rem}} \times \mathbb{D}^2$ —a disjoint union of solid tori—along the boundary components of $\partial_v E'$ that touch $\partial_h^{\text{rem}} E$. Again this attachment has nothing to do with the region $E' \setminus E$, so we can define

$$\partial_v \tilde{E} := \partial_v E \cup (\partial \Sigma^{\text{rem}} \times \mathbb{D}^2) \subset \partial_v \tilde{E}'.$$

Each of these gives rise to collars which are also subsets of \tilde{E}' : we shall denote

$$\mathcal{N}(\partial_h \tilde{E}') := \mathcal{N}(\partial_h \tilde{E}) := (-1, 0] \times \partial_h^{\text{oth}} E = (-1, 0] \times \Sigma^{\text{oth}} \times S^1 \subset \tilde{E},$$

with t denoting the coordinate in $(-1, 0]$, and

$$\begin{aligned} \mathcal{N}(\partial_v \tilde{E}') &:= (-1, 0] \times \partial_v \tilde{E}' \subset \tilde{E}', \\ \mathcal{N}(\partial_v \tilde{E}) &:= (-1, 0] \times \partial_v \tilde{E} \subset \tilde{E}, \end{aligned}$$

with the coordinate on $(-1, 0]$ denoted by s . These collars do not cover all of \tilde{E}' , as we also have

$$\tilde{\mathcal{N}}(\partial_h^{\text{rem}} E) := \mathcal{N}(\partial_h^{\text{rem}} E) \cup_{\Psi} (\Sigma^{\text{rem}} \times \mathbb{D}^2) \cong \Sigma^{\text{rem}} \times \mathbb{D}^2 \subset \tilde{E};$$

see Figures 10 and 11. Here the slightly different notational convention is meant to emphasize the fact that $\tilde{\mathcal{N}}(\partial_h^{\text{rem}} E)$ is not actually a collar neighborhood of any part of the boundary—it contains the original $\partial_h^{\text{rem}} E$, but this now lives in the *interior* of the “handle” $\Sigma^{\text{rem}} \times \mathbb{D}^2$. With this notation in place, \tilde{E}' and \tilde{E} can be presented as the unions of overlapping regions

$$\begin{aligned} \tilde{E}' &= \mathcal{N}(\partial_v \tilde{E}') \cup \mathcal{N}(\partial_h \tilde{E}) \cup \tilde{\mathcal{N}}(\partial_h^{\text{rem}} E), \\ \tilde{E} &= \mathcal{N}(\partial_v \tilde{E}) \cup \mathcal{N}(\partial_h \tilde{E}) \cup \tilde{\mathcal{N}}(\partial_h^{\text{rem}} E). \end{aligned}$$

The fibrations $\Pi_v : \mathcal{N}(\partial_v E) \rightarrow (-1, 0] \times S^1$ and $\Pi_h : \mathcal{N}(\partial_h E) \rightarrow \Sigma$ extend in obvious ways: on the horizontal neighborhoods we have trivial projections

$$\begin{aligned} \tilde{\Pi}_h : \mathcal{N}(\partial_h \tilde{E}) &= (-1, 0] \times \Sigma^{\text{oth}} \times S^1 \rightarrow \Sigma^{\text{oth}}, \\ \tilde{\Pi}_h : \tilde{\mathcal{N}}(\partial_h^{\text{rem}} E) &\cong \Sigma^{\text{rem}} \times \mathbb{D}^2 \rightarrow \Sigma^{\text{rem}}, \end{aligned}$$

and on the vertical collars, the formula $\Pi_v(s, \phi, t, \theta) = (s, m\phi)$ produces an extension

$$\tilde{\Pi}_v : \mathcal{N}(\partial_v \tilde{E}) \rightarrow (-1, 0] \times S^1$$

which is defined on each connected component of the attached region $(-1, 0] \times \partial \Sigma^{\text{rem}} \times \mathbb{D}^2$ by

$$(-1, 0] \times S^1 \times \mathbb{D}^2 \xrightarrow{\tilde{\Pi}_v} (-1, 0] \times S^1 : (s, \phi, \zeta) \mapsto (s, m\phi),$$

with the multiplicity $m \in \mathbb{N}$ as usual depending on the component under consideration. Denote the resulting vertical subbundle by $V\tilde{E} \subset T\tilde{E}$ and observe that

$$\tilde{\omega}_K|_{V\tilde{E}} > 0$$

by construction.

It will be useful to decompose $\partial_v\tilde{E}$ and $\partial_v\tilde{E}'$ further into the components

$$\partial_v\tilde{E} = \partial_v^{\text{flat}}\tilde{E} \amalg \partial_v^{\text{cvx}}\tilde{E}, \quad \partial_v\tilde{E}' = \partial_v^{\text{flat}}\tilde{E}' \amalg \partial_v^{\text{cvx}}\tilde{E}',$$

with corresponding collars $\mathcal{N}(\partial_v^{\text{flat}}\tilde{E})$, $\mathcal{N}(\partial_v^{\text{cvx}}\tilde{E})$ and $\mathcal{N}(\partial_v^{\text{cvx}}\tilde{E}')$, where $\partial_v^{\text{flat}}\tilde{E}$ is defined as the union of all components of $\partial_v\tilde{E}$ such that the fibers of $\tilde{\Pi}_v : \mathcal{N}(\partial_v^{\text{flat}}\tilde{E}) \rightarrow (-1, 0] \times S^1$ have empty boundary. Such components arise whenever M_P has components with boundary contained in $\pi_{\Sigma}^{-1}(\Sigma^{\text{rem}})$; see Figure 11. The notation is motivated by the fact that, as we'll see below, $\partial_v^{\text{cvx}}\tilde{E}'$ inherits a natural contact structure that is dominated by $\tilde{\omega}_K$, hence making $\partial_v^{\text{cvx}}\tilde{E}'$ weakly convex, but $\partial_v^{\text{flat}}\tilde{E}$ does not; in fact for certain natural choices of almost complex structure on \tilde{E} , $\partial_v^{\text{cvx}}\tilde{E}'$ is pseudoconvex while $\partial_v^{\text{flat}}\tilde{E}$ is Levi flat.

The fibrations $\tilde{\Pi}_v$ and $\tilde{\Pi}_h$ induce on $\partial_v^{\text{cvx}}\tilde{E} \cup \partial_h\tilde{E}$ the structure of a spinal open book $\tilde{\pi}$ with paper $\partial_v^{\text{cvx}}\tilde{E}$ and spine $\partial_h\tilde{E}$, the latter fibering over Σ^{oth} .

Lemma 4.4. *After smoothing the corner of $\partial_v^{\text{cvx}}\tilde{E} \cup \partial_h\tilde{E}$, $\tilde{\pi}$ supports a contact structure that is dominated by $\tilde{\omega}_K$.*

Proof. We mimic the procedure that was used in §4.1 to define λ_K on E' : first choose a fiberwise Liouville form $\tilde{\lambda}$ on $\partial_v^{\text{cvx}}\tilde{E}$ that equals $e^t d\theta$ in the collar neighborhoods of remaining boundary components (i.e. those that were not capped off in the transformation from $\partial_v E$ to $\partial_v^{\text{cvx}}\tilde{E}$). Pulling back via the obvious projection defines $\tilde{\lambda}$ on $\mathcal{N}(\partial_v^{\text{cvx}}\tilde{E})$, and the formula $\tilde{\lambda} = e^t d\theta$ extends it over $\mathcal{N}(\partial_h\tilde{E})$. (Note that by Stokes' theorem, $\tilde{\lambda}$ cannot be extended to $\mathcal{N}(\partial_v^{\text{flat}}\tilde{E})$.) We can then use the Thurston trick to define a Liouville form

$$\tilde{\lambda}_K := K\sigma + \tilde{\lambda}$$

on $\mathcal{N}(\partial_v^{\text{cvx}}\tilde{E}) \cup \mathcal{N}(\partial_h\tilde{E})$, after possibly increasing the value of $K > 0$, and this Liouville form matches λ_K on the regions where $\tilde{\lambda} = e^t d\theta$ and can thus be extended to $\mathcal{N}(\partial_v\tilde{E}) \setminus \mathcal{N}(\partial_v^{\text{flat}}\tilde{E})$ in the same way as λ_K . We claim now that if $K > 0$ is sufficiently large, then

$$\tilde{\lambda}_K \wedge \tilde{\omega}_K|_{T(\partial_v^{\text{cvx}}\tilde{E}')} > 0 \quad \text{and} \quad \tilde{\lambda}_K \wedge \tilde{\omega}_K|_{T(\partial_h\tilde{E}')} > 0.$$

The second relation is immediate because $\lambda_K = \tilde{\lambda}_K$ near $\partial_h\tilde{E}'$, so we are merely rephrasing the fact that $\partial_h^{\text{oth}}E$ is a contact hypersurface in $(E, d\lambda_K)$. The first relation is similarly immediate on the regions where $\lambda_K = \tilde{\lambda}_K$, so we only still need to check that it holds on $\partial_v^{\text{cvx}}\tilde{E}$. To see this, notice that $\tilde{\omega}_K$ can be written in $\mathcal{N}(\partial_v^{\text{cvx}}\tilde{E})$ as

$$\tilde{\omega}_K = Kd\left(e^s d\tilde{\Pi}_v\right) + \omega_{\text{fib}},$$

where ω_{fib} is a closed 2-form that satisfies $\omega_{\text{fib}}|_{V\tilde{E}} > 0$ and is independent of K , while $e^s d\tilde{\Pi}_v$ can be regarded as the pullback via $\tilde{\Pi}_v$ of a Liouville form on $[-1, 0] \times S^1$. The claim thus follows via Proposition 2.4.

Finally, the same argument used previously for λ_K shows that $\tilde{\lambda}_K$ restricts to both $\partial_h\tilde{E}'$ and $\partial_v^{\text{cvx}}\tilde{E}'$ as a contact form, and by construction it matches the contact form induced by

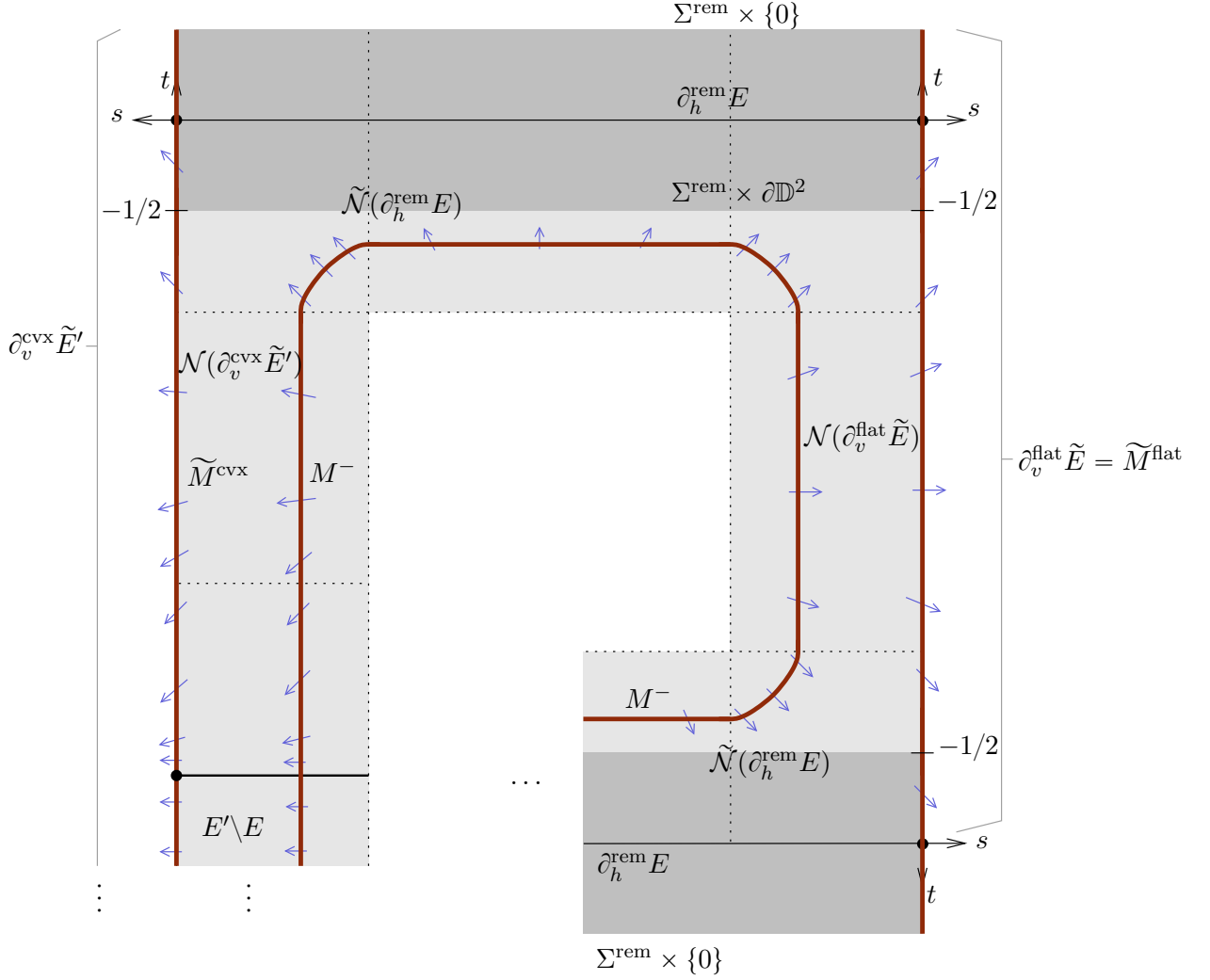


FIGURE 11. A variant of Figure 10 in which $\Sigma^{\text{rem}} \times \mathbb{D}^2$ has two connected components, attached at both the top and the bottom of the picture. The upper boundary of the cobordism now includes a component \tilde{M}^{flat} that is not contact, as it is foliated by closed pages of a *generalized* spinal open book. (Note that the only actual boundary of \tilde{E}' in this picture is at the sides; the top and bottom represent two distinct connected components of the *interior* submanifold $\Sigma^{\text{rem}} \times \{0\} \subset \Sigma^{\text{rem}} \times \mathbb{D}^2$.)

giving rise in this case to a weakly contact hypersurface

$$(\tilde{M}^{\text{cvx}}, \tilde{\xi}) \subset (\tilde{E}', \tilde{\omega}_K)$$

whose contact structure $\tilde{\xi}$ is defined by restricting $\tilde{\lambda}_K$ to \tilde{M}^{cvx} . □

The weakly contact hypersurface $(\widetilde{M}^{\text{cvx}}, \widetilde{\xi})$ found in the above proof is shown in Figure 10 as the smooth curve traversing the outer boundary of \widetilde{E}' with some rounding at the corners. We are now in a position to define an actual spine removal cobordism: let

$$X \subset \widetilde{E}'$$

denote the region that is sandwiched in between $M^- \subset E' \subset \widetilde{E}'$ and $\partial_v^{\text{flat}} \widetilde{E} \amalg \widetilde{M}^{\text{cvx}} \subset \widetilde{E}'$, making $(X, \widetilde{\omega}_K)$ a compact symplectic manifold with strongly concave boundary (M^-, ξ_-) , weakly convex boundary $(\widetilde{M}^{\text{cvx}}, \widetilde{\xi})$, and additional boundary components $\partial_v^{\text{flat}} \widetilde{E}$ which are neither concave nor convex but are fibered by closed symplectic surfaces.

To finish the proof of Theorem 1.25, we need to modify $(X, \widetilde{\omega}_K)$ to allow symplectic forms that are not exact at the negative boundary. Suppose Ω is a closed 2-form on M' that satisfies $\Omega|_{\xi} > 0$ and is exact on $\Sigma^{\text{rem}} \times S^1$. We can then find a closed 2-form η on M' with the following properties:

- (1) $[\eta] = [\Omega] \in H_{\text{dR}}^2(M')$;
- (2) On each of the collar components $S^1 \times (-1, 0] \times S^1 \subset \mathcal{N}(\partial M_P)$ and $(-1, 0] \times S^1 \times S^1 \subset \mathcal{N}(\partial M_\Sigma)$, η is a constant multiple of $d\phi \wedge d\theta$;
- (3) η vanishes on $\pi_\Sigma^{-1}(\Sigma^{\text{rem}}) = \Sigma^{\text{rem}} \times S^1$.

The third condition is possible due to the cohomological assumption, and combining this assumption with the second condition implies that η also vanishes on all components of $\mathcal{N}(\partial M_P)$ adjacent to $\pi_\Sigma^{-1}(\Sigma^{\text{rem}})$. We can now define η as a closed 2-form on $\mathcal{N}(\partial_v E')$ and $\mathcal{N}(\partial_h E')$ by pulling back via the projections $(-1, 0] \times M'_P \rightarrow M'_P$ and $(-1, 0] \times M'_\Sigma \rightarrow M'_\Sigma$ respectively, and the second condition implies that η remains well defined after gluing these collars together to form E' , thus we shall regard η as a closed 2-form on E' . By construction, η vanishes near $\partial_h^{\text{rem}} E$, hence η can also be regarded as defining a closed 2-form on \widetilde{E}' . Its restriction

$$\eta^- := \eta|_{TM^-}$$

is cohomologous to Ω after identifying M^- with M . The following is an immediate consequence of the fact that the nondegeneracy of 2-forms and the “weakly contact” condition are both open.

Lemma 4.5. *There exists a constant $C_0 > 0$ such that for all $C \geq C_0$, the 2-form*

$$\widetilde{\omega}'_K := C\widetilde{\omega}_K + \eta$$

is symplectic on X , the boundary components (M^-, ξ_-) and $(\widetilde{M}^{\text{cvx}}, \widetilde{\xi})$ are weakly concave and convex respectively, and $\widetilde{\omega}'_K$ is positive on the closed surface fibers in $\partial_v^{\text{flat}} \widetilde{E}$. \square

Finally, observe that since Ω and η are cohomologous on M' and $\Omega|_{\xi} > 0$, [MNW13, Lemma 2.10] provides a symplectic form on $[0, 1] \times M'$ that restricts to Ω on $\{0\} \times M'$ and $C d\alpha' + \eta$ on $\{1\} \times M'$, where one has the freedom to choose α' as any contact form for ξ at the expense of inserting a sufficiently large constant $C > 0$. We can therefore make these choices and increase the value of $C \geq C_0$ if necessary so that the weak symplectic cobordism $(X, \widetilde{\omega}'_K)$ provided by Lemma 4.5 can be attached on top of $[0, 1] \times M'$. All together, this provides a weak symplectic cobordism with the properties stated in Theorem 1.25 and thus completes the proof.

5. NONFILLABILITY VIA SPINE REMOVAL

In this section we use spine removal surgery to prove Theorems 1.29 and 1.34. Theorem 1.29 will be an immediate corollary of the following result, using the method of [ABW10]; it says essentially that any contact manifold with a partially planar domain can be given a *symplectic cap* that contains a nonnegative symplectic sphere.

Theorem 5.1. *Suppose (M', ξ) is a contact 3-manifold containing an Ω -separating partially planar domain for some closed 2-form Ω with $\Omega|_\xi > 0$. Then there exists a compact symplectic manifold (X, ω) with $\partial X = -M'$ and $\omega|_{TM'} = \Omega$ such that (X, ω) contains a symplectically embedded 2-sphere with vanishing self-intersection number.*

Proof. Let $M \subset M'$ denote the partially planar domain, $M_P^{\text{pln}} \subset M$ its planar piece, and $\Sigma_1 \times S^1, \dots, \Sigma_r \times S^1 \subset M_\Sigma$ the smallest collection of spinal components that contain $\partial M_P^{\text{pln}}$. Since Ω is exact on all these components, Theorem 1.25 provides a spine removal cobordism (X_0, ω) with

$$\partial X_0 = -M' \amalg \widetilde{M}'$$

and $\omega|_{TM'} = \Omega$, constructed by attaching handles $\Sigma_i \times \mathbb{D}^2$ along each of the spinal components surrounding $\partial M_P^{\text{pln}}$. The surgered manifold \widetilde{M}' is then disconnected and can be written as

$$\widetilde{M}' = \widetilde{M}'_1 \amalg \widetilde{M}'_2,$$

where \widetilde{M}'_1 is a symplectic sphere bundle over S^1 , and \widetilde{M}'_2 is either a contact manifold $(\widetilde{M}'_2, \xi_2)$ with $\omega|_{\xi_2} > 0$ or another symplectic fibration over S^1 with closed fibers. Both components can now be capped using the method of Eliashberg [Eli04], and the symplectic S^2 -fibers of \widetilde{M}'_1 give the desired symplectic spheres with vanishing self-intersection. \square

We recall briefly why this result implies Theorem 1.29: if (W, ω) is a closed symplectic 4-manifold and $M \hookrightarrow W$ is a (weak) contact embedding that does not separate W , then by cutting W open along M we obtain a (weak) symplectic cobordism between (M, ξ) and itself. Attaching infinitely many copies of this cobordism to each other in a sequence, one constructs a “noncompact symplectic filling” $(W_\infty, \omega_\infty)$ of (M, ξ) which is nonetheless geometrically bounded. If (M, ξ) contains a partially planar domain for which ω_∞ is exact on the spine, then one can attach the cap from Theorem 5.1 and then choose a geometrically bounded compatible almost complex structure J_∞ so that the symplectic spheres in the cap become embedded J_∞ -holomorphic spheres which are Fredholm regular and have index 2. Arguing as in McDuff [McD90], the moduli space generated by these spheres is then compact and foliates all of W_∞ , but this is impossible since the latter is noncompact. The full details for the case $[\Omega] = 0 \in H_{\text{dR}}^2(M)$ are carried out in [ABW10], and the generalization for nontrivial cohomology classes following the above scheme is immediate.

For planar torsion, we will make use of the following simple lemma in the style of [McD90]:

Lemma 5.2. *Suppose (W, ω) is a compact symplectic 4-manifold, possibly with boundary, such that ∂W carries a positive contact structure dominated by ω . Suppose moreover that W contains a symplectically embedded sphere $S_1 \subset W$ with vanishing self-intersection number. Then $\partial W = \emptyset$, and any other symplectically embedded surface $S_2 \subset W \setminus S_1$ with vanishing self-intersection is also a sphere and satisfies*

$$\int_{S_1} \omega = \int_{S_2} \omega.$$

Proof. Choose a compatible almost complex structure J which preserves the contact structure at the boundary and makes both S_1 and S_2 J -holomorphic. Then S_1 is a Fredholm regular index 2 curve, and arguing as in [McD90], we find that the set of all J -holomorphic curves homotopic to S_1 foliates W except at finitely many nodal singularities, which are intersections of finitely many J -holomorphic exceptional spheres. Then if $\partial W \neq \emptyset$, some holomorphic sphere must touch ∂W tangentially, thus violating J -convexity. Moreover, positivity of intersections implies that no curve in this family can have any isolated intersection with S_2 , thus S_2 itself must belong to the family, implying that it is a sphere with the same symplectic area as S_1 . \square

Proof of Theorem 1.34. Consider again the spine removal cobordism (X_0, ω) from the proof of Theorem 5.1 with $\partial X_0 = -M' \amalg \widetilde{M}'$ and $\widetilde{M}' = \widetilde{M}'_1 \amalg \widetilde{M}'_2$, but now under the extra assumption that the partially planar domain $M \subset M'$ is not symmetric. This implies in particular that in addition to the planar piece M_P^{pln} , the paper $M_P \subset M$ contains another connected component $M_P^{\text{oth}} \subset M_P$ for which at least one of the following is true:

- (1) $\partial M_P^{\text{oth}}$ is not contained in $\Sigma_1 \times S^1 \cup \dots \cup \Sigma_r \times S^1$;
- (2) The pages in M_P^{oth} have positive genus;
- (3) The pages in M_P^{oth} have genus zero but there is a spinal component $\Sigma_i \times S^1$ that contains differing numbers of boundary components of pages in M_P^{pln} and M_P^{oth} .

In the first case, it follows that \widetilde{M}'_2 carries a contact structure dominated by ω , so after capping \widetilde{M}'_1 we have a contradiction to Lemma 5.2. In the second case, either the same thing happens or \widetilde{M}'_2 is a symplectic fibration over S^1 with closed pages of positive genus, so capping both \widetilde{M}'_1 and \widetilde{M}'_2 with Lefschetz fibrations as in [Eli04] gives disjoint symplectically embedded surfaces with zero self-intersection, one rational and one not, again contradicting the lemma. For the third case we instead may obtain two disjoint symplectically embedded spheres, but they can be arranged to have different symplectic area. \square

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