More Fraïssé limits of nilpotent groups of finite exponent

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Abstract

The class of nilpotent groups of class c and prime exponent $p > c$ with additional predicates $P_c \subseteq P_{c-1} \subseteq \ldots \subseteq P_1$ for suitable subgroups has the amalgamation property. Hence the Fraïssé limit D of the finite groups of this class exists. $\langle 1 \rangle \subseteq P_c(D) \subseteq \ldots \subseteq P_2(D) \subseteq P_1(D) = D$ is the lower and the upper central series of D. In this extended language D is ultrahomogeneous. The elementary theory of D allows the elimination of quantifiers and it is $\aleph_0$-categorical. For $c = 2$ this was proved in [2].

AMS Classification: 03 C 60, 03 C 98

1 Introduction

We generalize the results of the first part of [2]. Let $\mathfrak{G}_{c,p}$, where $c < p$ and p prime, be the variety of the nilpotent groups of class c and of exponent p. As in the case $c = 2$ ([2]) it is necessary to extend the language L of group theory to prove the amalgamation property.

Let $\langle 1 \rangle = \Gamma_{c+1}(G) \subseteq \Gamma_c(G) \subseteq \ldots \subseteq \Gamma_2(G) \subseteq \Gamma_1(G) = G$ be the lower central series and $Z_0(G) = \langle 1 \rangle \subseteq Z_1(G) \subseteq \ldots \subseteq Z_c(G) = G$ be the upper central series of $G \in \mathfrak{G}_{c,p}$. If $X \subseteq G$, then let $\langle X \rangle$ be the subgroup of G generated by X. In general $\langle X \rangle$ is the substructure of the structure G generated by X. $[X, Y]$ is used to denote $\{[x, y] : x \in X,\ y \in Y\}$, where $[x, y] = x^{-1}y^{-1}xy$. Let G' be the commutator subgroup $\langle [G, G] \rangle$.

Definition Let G be a group in $\mathfrak{G}_{c,p}$. We often write Q_n, P_n, Z_n instead of $Q_n(G)$, $P_n(G)$, and $Z_n(G)$ respectively. Let Q_n and P_n be subgroups of G such that

$Q_1 = P_1 = G$ and for $1 < n \leq c + 1$ we have $Q_n = \left\langle \bigcup_{\ell+k=n} [P_\ell, P_k] \right\rangle$ and

P_n is a subgroup of G with $Q_n \subseteq P_n \subseteq Z_{c+1-n}$ and $P_n \subseteq P_{n-1}$.

Let $\mathfrak{G}^P_{c,p}$ be the class of all groups in $\mathfrak{G}_{c,p}$ with additional predicates $P_c(G) \subseteq \ldots \subseteq P_2(G)$ as described above. Let L^+ be the corresponding language that extends L.

Note that $P_n(G)$ is a normal subgroup of G. It is easily seen that $Q_n(G) \subseteq Q_{n-1}(G)$. We also use $Q_1(G) = P_1(G) = G$ and $Q_{c+1}(G) = P_{c+1}(G) = \langle 1 \rangle$. Furthermore $[G, P_n] \subseteq Q_{n+1} \subseteq P_{n+1}$. Hence P_n/P_{n+1} is abelian; in fact central in G/P_{n+1}. The structures in $\mathfrak{G}^P_{c,p}$ we often call L^+-groups.

Note that every group $G \in \mathfrak{G}_{c,p}$ has an L^+-expansion in $\mathfrak{G}^P_{c,p}$. We define $P_n(G) = Q_n(G) = \Gamma_n(G)$.

As in [2] we use R. Fraïssé's construction [3]. We follow the presentation of W. Hodges in ([4], Theorem 7.1.2). The main result of this paper is the following:

Theorem 1.1 *$\mathfrak{G}^P_{c,p}$ has the amalgamation property* (AP).

Then (AP) implies the joint embedding property (JEP) of $\mathfrak{G}^P_{c,p}$. Furthermore $\mathfrak{G}^P_{c,p}$ has the herditary property (HP): Every substructure of a L^+-group in $\mathfrak{G}^P_{c,p}$ is again in $\mathfrak{G}^P_{c,p}$. Note that (HP) implies that a definition $P_n(G) = \Gamma_n(G)$ for all $G \in \mathfrak{G}^P_{c,p}$ is impossible. By R. Fraïssé's result quoted above we have:

Corollary 1.2 *The Fraïssé limit D of the finite L^+-groups in $\mathfrak{G}^P_{c,p}$ exists. Every countable or finite group in $\mathfrak{G}_{c,p}$ can be embedded into $D \restriction L$.*

Corollary 1.3 *For the Fraïssé limit D we have that $\Gamma_n(D) = Q_n(D) = P_n(D) = Z_{c+1-n}(D)$ for $1 \leq n \leq c$. D is ultrahomogeneous in the language L^+. The elementary L^+-theory of D with this interpretation of the predicates P_n allows the elimination of quantifiers.*

Corollary 1.4 *The elementary theories of D in L and L^+ are $\aleph_0$-categorical.*

In [2] the well-known functor from $\mathfrak{G}^P_{2,p}$ into the category of alternating bilinear maps is used to prove Theorem 1.1 in the case $c = 2$. For $\mathfrak{G}^P_{c,p}$ with $c > 2$ the corresponding functor into the category of graded Lie-algebras loses some information. Therefore we need a new strategy to prove the result:

Let $F_c(p, \kappa)$ be the free group in $\mathfrak{G}_{c,p}$ with κ free generators. In [1] the subgroups F of $F_c(p, \kappa)$ are investigated. Their algebraic structure is very similar to the structure of $F_c(p, \kappa)$. For $F \subseteq F_c(p, \kappa)$ we can define $P_n(F) = Z_{c+1-n}(F_c(p, \kappa)) \cap F$. The corresponding L^+-structure we denote again by F. We call it a quasi-free group in $\mathfrak{G}^P_{c,p}$. For $G \in \mathfrak{G}^P_{c,p}$ we call F a quasi-free group with the same dimensions as G, if

$$\dim(P_n(G)/\langle P_{n+1}(G) \cup Q_n(G)\rangle) = \dim(P_n(F)/\langle P_{n+1}(F) \cup Q_n(F)\rangle).$$

From results in [1] follows that the isomorphism type of F is fixed by G via the condition above. We show that there is a strong L^+-homomorphism f of F onto G. f is strong if $P_n(f(a))$ in G implies $P_n(b)$ in F for some b in F with $f(a) = f(b)$. Let N be the normal subgroup of F such that $G \cong F/N$. We can show that the isomorphism type of the pair (F, N) is uniquely determined by G. We call (F, N) the canonical pair for G. These canonical pairs are the main tool in the proof of Theorem 1.1.

In the next section we prepare results of [1] for the purposes of this paper. In Section 3 we prove (AP) for $\mathbb{G}^P_{c,p}$.

In [2] the results are proved for $c = 2$. There non-forking is characterized for $\mathrm{Th}(D)$. It does not have symmetry and transitivity. I want to mention here, that also independence over a model fails.

2 Subgroups of $F_c(p, \kappa)$

All results which are stated here you will find in [1], Section 3. The background is the work of P. Hall, W. Magnus, E. Witt, and M. Hall. But the main result used in [1] is the theorem of Širšov-Witt that every subalgebra of a free Lie algebra over a field is again free.

We use $\mathbb{Z}$ to denote the ring of the integers. All considered groups G are nilpotent of class c and of exponent p.

Let A be a subset of G. The elements of A are basic commutators of A-weight 1. By induction on the A-weight n we define basic commutators b of A-weight $w_A(b) = n$ on A. Every definition of basic commutators involves a choice of total order $<$ on them, such that $w_A(b_1) < w_A(b_2)$ implies $b_1 < b_2$. Hence the elements of A are the smallest elements in this order. Now we assume that the basic commutators on A of A-weight less than n have already been defined and ordered. The A-weight of a basic commutator b is n if $b = [b_1, b_2]$ where b_1 and b_2 are basic commutators, $w_A(b_1)+w_A(b_2) = n$, $b_1 > b_2$, and if $b_1 = [b_3, b_4]$, then $b_2 \geq b_4$.

There are several procedures to obtain basic commutators on A depending on the order chosen for each A-weight. If we speak about basic commutators over A, then we assume one fixed choice. We assume that the signature of L is $\{\cdot, {}^{-1}, 1\}$. The following fact is well-known:

Fact 2.1 *For every term $t(x_1, \ldots, x_n)$ of L there is a sequence $y_0(x_1, \ldots, x_n) < \ldots < y_{m-1}(x_1, \ldots, x_n)$ of basic commutators over $\{x_1, \ldots, x_n\}$ such that*

$$\mathbb{G}_{c,p} \models t(x_1, \ldots, x_n) = \prod_{i<m} y_i(x_1, \ldots, x_n)^{r_i}$$

where $0 \leq r_i < p$. We get the equality using only identities of the form $cd = dc[c, d]$ and $c^p = 1$.

To prove Fact 2.1 we can use the following procedure. Let g be $t(x_1, \ldots, x_n)$.

After n steps we have $g = y_1^{r_1} \ldots y_k^{r_k} g_1 \ldots g_\ell$ where the y_i and g_j are basic commutators, $y_1 < y_2 < \ldots < y_k$ and $y_k < g_j$ for $1 \leq j \leq \ell$. (At the beginning there is no y_i and $g_j \in \{x_1, \ldots, x_n\}$. x_i^{-1} is represented by a product of $(p-1)$ many x_i.) Let

$g_{i_1} = g_{i_2} = \ldots = g_{i_t}$ with $i_1 < i_2 < \ldots < i_t$ be the smallest basic commutator on the right side. Using $cd = dc[c,d]$ we move first g_{i_1} then g_{i_2} and so on to the place after $y_k^{r_k}$. By this procedure we produce only basic commutators of $\{x_1, \ldots, x_n\}$-weight greater than $w_{\{x_1,\ldots,x_n\}}(y_k)$. Since in $\mathfrak{G}_{c,p}$ commutators of weight $> c$ are 1 we can stop if $w_{\{x_1,\ldots,x_n\}}(g_j) > c$ for all g_j.

Now we consider the free groups $F_c(p,\kappa)$ in $\mathfrak{G}_{c,p}$.

Theorem 2.2 (*Second Basis Theorem of P. Hall*) *Let $A = \{a_\alpha : \alpha < \kappa\}$ be a set of free generators of $F_c(p,\kappa)$. Then $\Gamma_i(F_c(p,\kappa))/\Gamma_{i+1}(F_c(p,\kappa))$ is a free module over the field $\mathbb{Z}/p\mathbb{Z}$ for $1 \le i \le c$. The cosets of all basic commutators of A-weight i form a basis of that module.*

Corollary 2.3 *Assume $\{b_\alpha : \alpha < \lambda\}$ is a sequence of all basic commutators on A ordered according to A-weight. Then every element g of $F_c(p,\kappa)$ can be uniquely expressed as*

$$g = \prod_\alpha b_\alpha^{r_\alpha} \qquad \text{where } r_\alpha \in \mathbb{Z}/p\mathbb{Z}$$

and $r_\alpha = 0$ up to finitely many α.

Corollary 2.4 $\Gamma_i(F_c(p,\kappa)) = Z_{c+1-i}(F_c(p,\kappa))$ *for* $1 \le i \le c+1$.

Theorem 2.2 provides us a free nilpotent Lie algebra $L[F_c(p,\kappa)]$ of class c over the field $\mathbb{Z}/p\mathbb{Z}$. As a $\mathbb{Z}/p\mathbb{Z}$-module let $L[F_c(p,\kappa)]$ be $\bigoplus_{1\le i\le c} L[F_c(p,\kappa)]^i$ where $L[F_c(p,\kappa)]^i = \Gamma_i(F_c(p,\kappa))/\Gamma_{i+1}(F_c(p,\kappa))$. For $\bar{a} = \sum_{1\le i\le c} \bar{a}_i$ and $\bar{b} = \sum_{1\le i\le c} \bar{b}$ where $\bar{a}_i, \bar{b}_i \in L[F_c(p,\kappa)]^i$ choose elements $a_i, b_i \in F_c(p,\kappa)$ in the cosets $\bar{a}_i$ resp. $\bar{b}_i$. We define $(\bar{a},\bar{b}) = \sum_{1\le i\le c} \bar{c}_i$ where $\bar{c}_i \in L[F_c(p,\kappa)]^i$ and $\bar{c}_i$ is the coset of $\prod_{r+s=i} [a_r, b_s]$. Then $(\bar{a},\bar{b})$ is a well-defined Lie multiplication. It is not hard to prove that $L[F_c(p,\kappa)]$ is a free nilpotent Lie algebra over $\mathbb{Z}/p\mathbb{Z}$ using Theorem 2.2 and similar results for free nilpotent Lie algebras.
Let G be a L^+-group in $\mathfrak{G}_{c,p}^P$ and $A \subseteq G$ where $A = \{a_\alpha^i : 1 \le i \le c\,, \alpha < \lambda_i\}$ and $a_\alpha^i \in P_i(G)$. We set $a_\alpha^i < a_\beta^j$ if $i < j$ or $i = j$ and $\alpha < \beta$. We define basic commutators over A as above starting with this order. In addition to A-weight we introduce the A-degree $d_A(b)$ of commutators b on A: $d_A(a_\alpha^i) = i$, $d_A([b_1,b_2]) = d_A(b_1) + d_A(b_2)$. Note that any commutator of A-degree i is in $P_i(G)$.

Definition Let G be a group in $\mathfrak{G}_{c,p}^P$ and $A = \{a_\alpha^i : 1 \le i \le c,\ \alpha < \lambda_i\} \subseteq G$. A is an (o)-system, if $a_\alpha^i \in P_i(G)$ for every i with $1 \le i \le c$, and $\{a_\alpha^i : \alpha < \lambda_i\}$ is linearly independent modulo the normal subgroup generated by $P_{i+1}(G)$ and all other basic commutators on A of A-degree i. A is a $(*)$-system, if $a_\alpha^i \in P_i(G)$ and the basic commutators over A of A-degree i are linearly independent modulo $P_{i+1}(G)$ for all i with $1 \le i \le c$.

In [1] these notions were defined for groups $G \in \mathfrak{G}_{c,p}$ with the property $[Z_{c+1-i}(G), Z_{c+1-j}(G)] \subseteq Z_{c+1-(i+j)}(G)$. If we define $P_i(G) = Z_{c+1-i}(G)$ for these groups, then the definitions in the two papers coincide. As in the introduction we call the L^+-subgroups F of the $F_c(p,\kappa)$ with $P_i(F) = Z_{c+1-i}(F_c(p,\kappa)) \cap F$ the *quasi-free groups of* $\mathfrak{G}^P_{c,p}$. Then a L^+-subgroup of a quasi-free group is quasi-free.
Analogously to [1] we have

Lemma 2.5 *Let G be in $\mathfrak{G}^P_{c,p}$.*

i) *For every subgroup of G there exists a generating (o)-system.*

ii) *Every $(*)$-system in G is an (o)-system.*

iii) *A is an (o)-system ($(*)$-system) in G if and only if every finite subset of A is an (o)-system ($(*)$-system) in G.*

In our terminology we can formulate Theorem 3.6 in [1]:

Theorem 2.6 *Let F be a quasi-free group. Then every (o)-system in F is a $(*)$-system.*

The proof of Theorem 2.6 is described in [1]. We consider $F \subseteq F_c(p,\kappa)$ and move to $L[F_c(p,\kappa)]$ to apply the key lemma behind the Theorem of Širšov-Witt.

Corollary 2.7 *Let F be a quasi-free group and $A = \{a^i_\alpha : 1 \leq i \leq c,\ \alpha < \lambda_i\}$ be an (o)-system in F. Let $\{b_\alpha : \alpha < \kappa\}$ be an enumeration of the basic commutators on A according to A-degree. Then every element g of $\langle A \rangle$ can be uniquely expressed as $g = \prod_\alpha b^{r_\alpha}_\alpha$ where $r_\alpha \in \mathbb{Z}/p\mathbb{Z}$ and $r_\alpha = 0$ for all but finitely many α. There is an algorithm to compute this representation of g using only identities of the form $cd = dc[c,d]$ and $c^p = 1$.*

Proof. By Theorem 2.6 A is a $(*)$-system. Let $\{b_\alpha : \alpha < \kappa_n\}$ be the set of all basic commutators on A of A-degree $\leq n$ for $1 \leq n \leq c$ with the given order. Let $\kappa_0 = 0$. We have $\kappa_c = \kappa$. We show the assertion for $g \in P_n$ by induction on n. Let g be an element of $P_n(F) \setminus P_{n+1}(F)$. Using only $ab = ba[a,b]$ we obtain

$$g = \prod_{\alpha<\kappa} c^{s_\alpha}_\alpha$$

where $\{c_\alpha : \alpha < n\}$ is an enumeration of the basic commutators on A according to A-weight (Fact 2.1).
Let $c_{\alpha_1} = b_{\beta_1}, \ldots, c_{\alpha_m} = b_{\beta_m}$ be the basic commutators in this representation of g of minmal A-degree t such that $s_{\alpha_i} \neq 0$. We have $\kappa_{t-1} \leq \beta_1 < \ldots < \beta_m < \kappa_t$. Let $r_{\beta_1} = s_{\alpha_1}, \ldots,$ and $r_{\beta_m} = s_{\alpha_m}$. Then $g = \prod_{1 \leq i \leq m} b^{r_{\beta_i}}_{\beta_i} g'$ where $g' \in P_{t+1}$. Since A is a $(*)$-system we have $t = n$. We obtain this representation using only identities $cd = dc[c,d]$ and $c^P = 1$. If we apply the induction hypothesis to g', the assertion follows. □

Corollary 2.8 *Let F be a quasi-free group and $A = \{a^i_\alpha : 1 \leq i \leq c,\ \alpha < \lambda_i\}$ an (o)-system in F. Then $\{a^i_\alpha : \alpha < \lambda_i\}$ is linearly independent modulo $\langle P_{i+1}(F) \cup Q_i(F) \rangle$ and $P_i(F)$ is generated by the basic commutators on A of A-degree $\geq i$.*

3 Canonical pairs and amalgamation

Let G and H be L^+-groups in $\mathfrak{G}^P_{c,p}$. f is a L^+-homomorphism of G into H, if f is a group homomorphism and $P_n(a)$ implies $P_n(f(a))$ for all $a \in G$ and for all $2 \leq n < c$. f is strong, if for every $a \in G$ with $P_n(f(a))$ there exists some $b \in G$ such that $P_n(b)$ and $f(b) = f(a)$.

Lemma 3.1 *Let F be a quasi-free group in $\mathfrak{G}^P_{c,p}$ and $G \in \mathfrak{G}^P_{c,p}$. Assume $A = \{a^i_\alpha : 1 \leq i \leq c,\ \alpha < \lambda_i\}$ is an (o)-system in F and $C = \{c^\alpha_i : 1 \leq i \leq c,\ \alpha < \lambda_i,\ c^\alpha_i \in P_i(G)\}$ is a subset of G. Then $f(a^i_\alpha) = c^i_\alpha$ extends to an L^+-homomorphism of $\langle A \rangle$ onto $\langle C \rangle$.*

Proof. By Theorem 2.6 A is a $(*)$-system. W.l.o.g. we can assume that A generates F. Let $\{b_\alpha : \alpha < \kappa\}$ be an enumeration of the basic commutators on A according to A-degree. Let d_α be the basic commutator over C that we obtain if we replace a^i_β by c^i_β in b_α. If b_α is in $P_n(F)$, then $d_A(b_\alpha) = n$ and d_α is in $P_n(G)$. If $g = \prod_\alpha b_\alpha^{r_\alpha}$ is the unique representation of g according to Corollary 2.7, then we define

$$f(g) = \prod_\alpha d_\alpha^{r_\alpha}\,.$$

By uniqueness of the representation of g we have that f is well-defined. If g is in $P_n(F)$, then all b_α with $r_\alpha \neq 0$ are in $P_n(F)$ since A is a $(*)$-system. Then all d_α with $r_\alpha \neq 0$ are in $P_n(G)$ and $f(g)$ is in $P_n(G)$.
It remains to show that f is a group homomorphism onto $\langle C \rangle$. Let $g = \prod_\alpha b_\alpha^{r_\alpha}$ and $h = \prod_\alpha b_\alpha^{s_\alpha}$ be two elements of $\langle A \rangle$. By Corollary 2.7 there is an algorithm to compute the unique representation of $g \cdot h = \prod_\alpha b_\alpha^{t_\alpha}$ using only identities of the form $cd = dc[c,d]$ and $c^p = 1$. If we apply the same steps to $f(g) \cdot f(h) = \prod_\alpha d_\alpha^{r_\alpha} \prod_\alpha d_\alpha^{s_\alpha}$, where a^i_α is replaced by c^i_α, then we obtain $f(g) \cdot f(h) = \prod_\alpha d_\alpha^{t_\alpha} = f(gh)$ as desired. Now it follows

$$f(a^{i_1}_{\alpha_1} a^{i_2}_{\alpha_2} \dots a^{i_n}_{\alpha_n}) = c^{i_1}_{\alpha_1} c^{i_2}_{\alpha_2} \dots c^{i_n}_{\alpha_n}.$$

Hence f is surjective onto $\langle C \rangle$. □

Definition Let G and H be groups in $\mathfrak{G}^P_{c,p}$. We define

$$\dim_i(G) = \dim(P_i(G)/\langle P_{i+1}(G) \cup Q_i(G)\rangle)\,.$$

We say that G and H have the same dimensions if

$$\dim_i(G) = \dim_i(H) \quad \text{for } 1 \leq i \leq c.$$

If F is a quasi-free group in $\mathfrak{G}^P_{c,p}$ and $A = \{a^i_\alpha : \alpha < \lambda_i,\ 1 \leq i \leq c\}$ is a generating (o)-system, then A is a $(*)$-system. Hence $\dim_i(F) = \lambda_i$. Using Theorem 3.12 in [1] we have

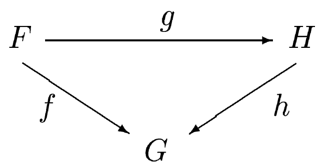
Lemma 3.2 *Quasi-free groups in $\mathfrak{G}^P_{c,p}$ with the same dimensions are isomorphic.*

Lemma 3.3 *Let G und F be groups in $\mathfrak{G}^P_{c,p}$, where F is quasi-free and* $\dim_i(G) = \dim_i(F)$ *for $1 \leq i \leq c$.*

i) *Let $A = \{a^i_\alpha : A \leq i \leq c,\ \alpha < \lambda_i\}$ be a generating (o)-system of F. Let f be a L^+-homomorphism of F onto G such that $\{f(a^n_\alpha) : \alpha < \lambda_n\}$ is a basis of $P_n(G)$ modulo $\langle P_{n+1}(G) \cup Q_n(G)\rangle$ for $1 \leq n \leq c$. Then f is strong.*

ii) *There is a strong L^+-homomorphism of F onto G.*

iii) *Let H be another quasi-free group with* $\dim_i(H) = \dim_i(G)$ *for $1 \leq i \leq c$. If f is a strong L^+-homomorphism of F onto G and h is a strong L^+-homomorphism of H onto G, then there is a L^+-isomorphism g of F onto H such that*



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Proof. i) We define $f(a^i_\alpha) = c^i_\alpha$ and $C = \{c^i_\alpha : 1 \leq i \leq c,\ \alpha < \lambda_i\}$. We have to show that f is strong. By induction on i we show:
If $e \in P_{c+1-i}(G)$, then there is some $a \in P_{c+1-i}(F)$ with $f(a) = e$. For $i = 0$ we obtain $e = 1$ and $a = 1$. Assume the assertion is true for $j < i$. Let $c + 1 - i = n$. Let $e \in P_n(G)$. By the assumption of i)

$$e = \prod_{1 \leq j \leq m} (c^n_{\alpha_j})^{r_j} q(e) w(e)$$

where $w(e) \in P_{n+1}(G)$, $q(e) \in Q_n(G)$. W.l.o.g. we can assume that $q(e)$ is a product of commutators of C-degree n over $\{c^j_\alpha : 1 \leq j < n,\ \alpha < \lambda_j\}$ since $q(e)$ is such a product modulo $P_{n+1}(G)$ by the commutator identities of Hall-Witt.
Let $q(a)$ be the element of F that we obtain if we replace c^j_α by a^j_α in $q(e)$. By induction there is some $w(a) \in P_{n+1}(F)$ with $f(w(a)) = w(e)$. If $a = \prod_{1 \leq j \leq m} (a^n_{\alpha_j})^{r_j} q(a) w(a)$, then $a \in P_n(F)$ and $f(a) = e$, as desired.
ii) Let $A = \{a^i_\alpha : 1 \leq i \leq c,\ \alpha < \lambda_i\}$ be a generating (o)-system of F. Then A is a $(*)$-system and $\lambda_i = \dim_i(F)$. Choose $C = \{c^i_\alpha : 1 \leq i \leq c,\ \alpha < \lambda_i\}$ such that $\{c^i_\alpha : \alpha < \lambda_i\}$ is a basis of $P_i(G)/\langle P_{i+1}(G) \cup Q_i(G)\rangle$ for $1 \leq i \leq c$. This is possible since $\dim_i(F) = \dim_i(G)$. By Lemma 3.1 there exists a L^+-homomorphism f of F onto $G = \langle C\rangle$ with $f(a^i_\alpha) = c^i_\alpha$. By i) f is strong.
iii) Choose a subset $E \subseteq G$ such that $E = \{e^i_\alpha : 1 \leq i \leq c,\ \alpha < \lambda_i\}$ and for $1 \leq i \leq c$ $\{e^i_\alpha : \alpha < \lambda_i\}$ is a basis for $P_i(G)/\langle P_{i+1}(G) \cup Q_i(G)\rangle$. Then $\lambda_i = \dim_i(G)$. For every pair (i, α) with $1 \leq i \leq c$ and $\alpha < \lambda_i$ choose $a^i_\alpha \in P_i(F)$ and $c^i_\alpha \in P_i(H)$ such that $f(a^i_\alpha) = e^i_\alpha = h(c^i_\alpha)$. This is possible since f and h are strong. Let A be $\{a^i_\alpha : 1 \leq i \leq$

$c,\ \alpha < \lambda_i\}$ and C be $\{c^i_\alpha : 1 \le i \le c,\ \alpha < \lambda_i\}$. Since $\dim_i(F) = \dim_i(G)$ for $1 \le i \le c$ we can show by induction on i that $\{a^i_\alpha : \alpha < \lambda_i\}$ is a basis of $P_i(F)/\langle P_{i+1}(F) \cup Q_i(F)\rangle$. Hence A is a generating (o)-system and therefore a $(*)$-system of F. Analogously C is a generating $(*)$-system of H.
Hence $g(a^i_\alpha) = c^i_\alpha$ provides a L^+-isomorphism by Lemma 3.2. Since $h(g(a^i_\alpha)) = e^i_\alpha = f(a^i_\alpha)$ we obtain the desired diagram. □

Definition Let N be a L^+-subgroup of $G \in \mathfrak{G}^P_{c,p}$ such that N is normal in G. Let $(G/N)^+$ be the L^+-group that we obtain from the L-factor group G/N if we define $P_i(aN)$ if there is some $b \in N$ with $P_i(ab)$ in G.

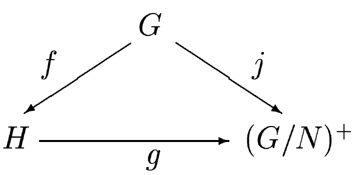
Note that $P_i(a_1b_1)$ and $P_i(a_2b_2)$ implies $P_i(a_1b_1a_2b_2)$ and $P_i(a_1a_2b_1[b_1,a_2]b_2)$ where $b_1[b_1,a_2]b_2$ is an element of N. Hence $P_i(G/N)$ is a subgroup of G/N. By definition $P_n((G/N)^+) \subseteq P_{n-1}((G/N)^+)$. If $P_n(aN)$, then $P_n(ab)$ for some $b \in N$. Hence $a \cdot b \in Z_{c+1-n}(G)$ and $aN \in Z_{c+1-n}(G/N)$. To show $Q_n((G/N)^+) \subseteq P_n((G/N)^+)$ let $aN \in P_\ell((G/N)^+)$ and $eN \in P_k((G/N)^+)$ where $\ell + k = n$. Then $ab \in P_\ell(G)$ and $ed \in P_k(G)$ for some $b, d \in N$. Hence $[ab, ed] \in P_n(G)$. Then $[ab, ed] = [a, e]u$ for some $u \in N$ and $[aN, bN] \in P_n((G/N)^+)$.

We have shown the first part of the following

Lemma 3.4 *Let G, H, N be in $\mathfrak{G}^P_{c,p}$.*

i) *If N is a normal L^+-subgroup of G, then $(G/N)^+$ is in $\mathfrak{G}^P_{c,p}$.*

ii) *Let f be a strong L^+-homomorphism of G onto H. Let N be the L-kernel of f. Define $P_n(N) = P_n(G) \cap N$. Let j be the canonical strong L^+-homomorphism of G onto $(G/N)^+$. Then there is a L^+-isomorphism g of H onto $(G/N)^+$ with*



Proof of ii) If we consider the situation in L, then the assertion is clear. Since f and j are strong the assertion follows. □

If $G \in \mathfrak{G}^P_{c,p}$, then Lemma 3.3ii) and 3.4 provides us a pair (F, N) such that

i) F is quasi-free,

ii) $\dim_i(F) = \dim_i(G)$ for $1 \le i \le c$, and

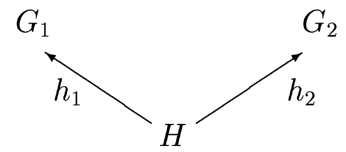
iii) N is a normal L^+-subgroup of F such that $G \underset{L^+}{\cong} (F/N)^+$.

By Lemma 3.3iii) the pair (F, N) is unique up to isomorphisms. We call (F, N) the canonical pair for G. Note that iii) can be replaced by the statement that there exists a strong L^+-homomorphism f of F onto G with kernel N.

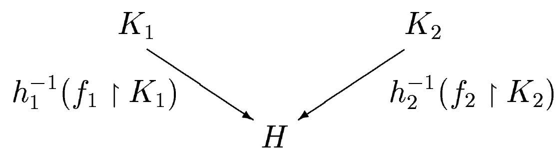
Lemma 3.5 *Assume $H \underset{L^+}{\subseteq} G$ and let (F, N) be the canonical pair for G. Then there is a L^+-subgroup K of F such that $(K, K \cap N)$ is the canonical pair for H.*

Proof. Let $f : F \to G$ be the strong L^+-homomorphism that corresponds to (F, N). We choose $E = \{e^i_\alpha : 1 \leq i \leq c,\ \alpha < \lambda_i\}$ in H such that $\{e^i_\alpha : \alpha < \lambda_i\}$ is a basis of $P_i(H)$ modulo $\langle P_{i+1}(H) \cup Q_i(H)\rangle$ for $1 \leq i \leq c$. For each pair (i, α) there is some $a^i_\alpha \in P_i(F)$ with $f(a^i_\alpha) = e^i_\alpha$ since f is strong. Let A be $\{a^i_\alpha : 1 \leq i \leq c,\ \alpha < \lambda_i\}$. Let K be $\langle A\rangle$ and $P_i(K) = K \cap P_i(F)$. Then $\{a^i_\alpha : \alpha < \lambda_i\}$ is a basis of $P_i(K)$ modulo $\langle P_{i+1}(K) \cup Q_i(K)\rangle$. Hence $\dim_i(K) = \dim_i(H)$. Lemma 3.3i) implies that $f \restriction K$ is strong. As a L^+-subgroup of F the L^+-group K is also quasi-free. This shows us, that $(K, K \cap N)$ is a canonical pair for H. □

Now we are ready to prove the amalgamation property for $\mathbb{G}^P_{c,p}$. Assume H can be L^+-embedded into G_1 and G_2, where all groups are in $\mathbb{G}^P_{c,p}$.



Let (F_i, N_i) be the canonical pair for G_i $(i = 1, 2)$ and f_i be the corresponding strong L^+-homomorphism of F_i onto G_i with kernel N_i. Furthermore we denote $h_i(H)$ by H_i. Then $H_i \underset{L^+}{\subseteq} G_i$. According to Lemma 3.5 we find $K_i \underset{L^+}{\subseteq} F_i$ such that $(K_i, K_i \cap N_i)$ is a canonical pair for H_i. We apply Lemma 3.3iii) to the situation



and obtain a L^+-isomorphism k of K_1 onto K_2 with the following property: If $f_1(v) = h_1(a)$ for $a \in H$ and $v \in K_1$, then $f_2(k(v)) = h_2(a)$.
Let F be the free product of F_1 and F_2 in $\mathbb{G}^P_{c,p}$. That means if $A = \{a^i_\alpha : 1 \leq i \leq c,\ \alpha < \lambda_i\}$ and $C = \{c^i_\alpha : 1 \leq i \leq c,\ \alpha < \kappa_i\}$ are generating $(*)$-systems of F_1 and F_2 respectively, then $A \cup C$ is a generating $(*)$-system of F.
Let J be the L^+-group that we obtain if we factorize F by the normal subgroup N where N is the smallest normal subgroup of F that contains $N_1 \cup N_2 \cup \{a(k(a))^{-1} : a \in K_1\}$. The following claim implies that J is an amalgam of G_1 and G_2 over H with respect to L^+.

Claim. $N \cap F_i = N_i \quad (i = 1, 2)$.

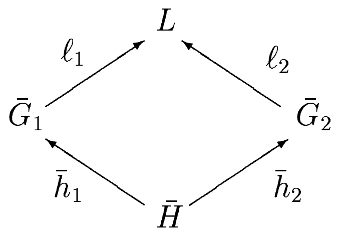
There is a well-known functor $G \rightsquigarrow \bar{G}$ from $\mathbb{G}^P_{c,p}$ into the class of c-nilpotent graded Lie algebras over the field $\mathbb{F}_p$ with p elements. (Using the Baker-Campbell-Hausdorff formula it can be shown that the functor is onto.)

$\bar{G}$ is build on $\bigoplus_{1 \leq i \leq c} P_i(G)/P_{i+1}(G)$ using the commutator to define the Lie multiplication as described for $G = F_c(p, w)$ and $P_i(G) = \Gamma_i(G)$ below Corollary 2.4. It is sufficient to prove the claim (and therefore the amalgamation) for the corresponding Lie algebras:

$$\bar{F}_i \cap \bar{N} = \bar{N}_i, \text{ where } \bar{N},\ \bar{N}_i \text{ are ideals in } \bar{F} \text{ and } \bar{F}_i \text{ respectively.}$$

We construct some Lie algebra L with embeddings ℓ_i of graded Lie algebras such that

(*)



By construction of $\bar{F}/\bar{N}$ (*) implies the same amalgamation with $\bar{F}/\bar{N}$ instead of L. Denote $P_i(G_1)/P_{i+1}(G_1)$ by $\bar{P}_i(G_1)$. To construct L let $\{e^i_\alpha : \alpha < \mu_i\}$ be a vector space basis of $\bar{P}_i(G_1)$ and $\{d^i_\alpha : \alpha < \nu_i\}$ be a vector space basis of $\bar{P}_i(G_2)$ $(1 \leq i \leq c)$ such that there are χ_i with

$$e^i_\alpha = d^i_\alpha \quad \text{for } \alpha < \chi_i \text{ and these elements form a basis of } \bar{H}.$$

Then we have equations

$$(**) \qquad [e^i_\alpha, e^j_\beta] = \sum_\gamma r^{i,j}_\gamma e^{i+j}_\gamma \quad \text{and} \quad [d^i_\alpha, d^j_\beta] = \sum_\gamma s^{i,j}_\gamma d^{i+j}_\gamma$$

that describe the Lie multiplication in $\bar{G}_1$ and $\bar{G}_2$ respectively. Let L be a vector space with the basis

$$\{e^i_\alpha : \alpha < \mu_i,\ 1 \leq i \leq c\} \cup \{d^j_\beta : \beta < \nu_i,\ 1 \leq j \leq c\}$$

with the identifications

$$e^i_\alpha = d^i_\alpha \quad \text{for } \alpha < \chi_i.$$

The Lie multiplication in L we define by $(**)$ and

$$[e^i_\alpha, d^j_\beta] = 0 \quad \text{for } \chi_i \leq \alpha \text{ and } \chi_j \leq \beta. \qquad \square$$

In the claim above N, F_i, and N_i are considered as L^+-subgroups of F. Especially $P_n(F) \cap F_i = P_n(F_i)$. The claim implies

$$F_i N/N \cong F_i/F_i \cap N \cong F_i/N_i \cong G_i.$$

This gives us a L-embedding j_i of G_i in $J = F/N$. Furthermore we have

$$N \cap P_n(F_i) = N \cap F_i \cap P_n(F) = N_i \cap P_n(F_i)$$

by the claim. Hence

$$(P_n(F) \cap F_i)N/N \cong P_n(F_i)/P_n(F_i) \cap N \cong P_n(F_i)/P_n(F_i) \cap N_i \cong P_n(G_i).$$

Therefore j_i is a L^+-embedding of G_i into $J = F/N$.

Let a be an element of H. We choose $v \in F_1$ as above such that $f_1(v) = h_1(a)$. Since $h_1(a) \in H_1$ we can choose v in K_1. By definition of k we have $f_2(k(v)) = h_2(a)$. Hence $j_1(h_1(a)) = vN$ and $j_2(h_2(a)) = k(v)N$. Since $vk(v)^{-1} \in N$ we have $j_1h_1(a) = j_2h_2(a)$ for all $a \in H$, as desired.

We have shown:

Theorem 1.1 *$\mathbb{G}^P_{c,p}$ has the amalgamation property.*

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