Stable Groups

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1 Examples

Stable groups are structures M where a group is a reduct of the structure and $\mathrm{Th}(M)$ is stable.

a) All infinite abelian groups are stable (Szmiliew).

b) Algebraic groups over algebraically closed fields. Fields.

c) Free groups (Sela).

d) Infinite free groups in the variety of c-nilpotent groups of exponent p^n with $p > c$ (B.).

e) Let L be a language of finite signature and T be a complete L-theory. Then there is a complete group theory T_G (Mekler) such that:

   T_G is a theory of 2-nilpotent groups of exponent p ($p > 2$);
   T is interpretable in T_G (stable embedding);
   T is κ-stable iff T_G is κ-stable;
   T is simple iff T_G is simple;
   T_G is not one-based;
   T is CM-trivial iff T_G is CM-trivial (B.).

f) Counterexamples to Zil'ber Conjecture in group theory: Uncountably categorical groups that are not one-based and do not allow the interpretation of an infinite field (B.). (They are 2-nilpotent of exponent $p > 2$.)

Cherlin-Zil'ber Conjecture: Every simple group of finite Morley rank is an algebraic group over an algebraically closed field.

Examples above are pure groups (no extra structure) or based on pure fields.

g) Examples with extra structure:

Black Field (Poizat/Baldwin, Holland): ACF_x with a predicate for a proper infinite subset of Morley rank 2.

Red Field (B., Martin-Pizarro, Ziegler): ACF_p with a predicate for a proper (red) infinite additive subgroup of Morley rank 2 ($p > 0$).

Bad Field (B., Hils, Martin-Pizarro, Wagner): ACF_0 with a predicate for a proper (green) infinite multiplicative subgroup of Morley rank 2.

Results about stable groups are often an important part of general stability theory.

2 Chain conditions

$$\begin{aligned} [g,h] &= g^{-1}h^{-1}gh \\ g^h &= h^{-1}gh \qquad g \rightsquigarrow g^h \quad (h \text{ fixed}) \text{ inner automorphism} \end{aligned}$$

G has exponent n, if n is minimal with $g^n = 1$ for all $g \in G$.

- Centralizer of A in G: $C_G(A) = \{g \in G : [g, A] = 1\}$
- Normalizer of H in G: $N_G(H) = \{g \in G : H^g = H\}$
- Derived series of G: $G^{(0)} = G$, $G^{(i+1)} = [G^{(i)}, G^{(i)}]$
- Lower central series: $\Gamma_0(G) = G$, $\Gamma_{n+1}(G) = [\Gamma_n(G), G]$.
  Note $\Gamma_1(G) = G^{(1)} = G'$.
- Upper central series: $Z_0(G) = \{1\}$, $Z_{n+1} = \{g \in G : [g,h] \in Z_n(G)$ for all $h \in G\}$.
- G is solvable of derived length n, if $G^{(n)} = \langle 1 \rangle$.
- G is nilpotent of class n if $\Gamma_{n+1}(G) = \langle 1 \rangle$ (iff $Z_n(G) = G$).

Definition The subgroups $H_1, H_2, \ldots$ of G are uniformly defined by $\varphi(x, \bar{y})$, if $H_i = \varphi_i(G, \bar{b})$ for some parameter $\bar{b}$.

Lemma 2.1 (Trivial Chain Condition) *Let G be a stable group. For every formula $\varphi(\bar{x}, \bar{y})$ there is some $n(\varphi)$ such that every chain $H_1 \subseteq H_2 \subseteq \ldots$ of subgroups uniformly defined by $\varphi(x, \bar{y})$ has length at most $n(\varphi)$.*

Lemma 2.2 *Let G be a stable group. For every formula $\varphi(x, \bar{y})$ there is some $m(\varphi)$ such that every intersection of a finite family of subgroups uniformly defined by $\varphi(x, \bar{y})$ is the intersection of at most $m(\varphi)$ of them.*

The *Proof* uses only NIP. Otherwise for each m there are $H_1 \dots H_m$ such that $H_i = \varphi(b, \bar{a}_i)$

$$\bigcap_{0 \le i \le m} H_i \subsetneq \bigcap_{\substack{0 < i \le m \\ i \ne j}} H_i \qquad \text{for every } \; j.$$

Choose b_j in the right but not in the left side.
If I is any subset of $\{1, \dots, m\}$, then define $b_I = \prod_{j \in I} b_j$. It follows $G \models \varphi(b_I, \bar{a}_i)$ iff $i \notin I$. □

Theorem 2.3 (Baldwin, Saxl) *Let G be a stable group. We have the chain condition for uniformly defined intersections subgroups of G.*

Proof. By Lemma 2.2 each element of a chain is uniformly defined as an intersection of at most $m(\varphi)$ φ-subgroups. Then apply Lemma 2.1. □

Example: Centralizers

Lemma 2.4 *If G is ω-stable, then G has no decreasing chain of definable subgroups.*

Proof. If $H_{i+1} \subsetneq H_i$ then

$$\mathrm{MR}(H_{i+1}) < \mathrm{MR}(H_i) \quad \text{ or } \quad \mathrm{MR}(H_{i+1}) = \mathrm{MR}(H_i)$$

but the Morley degree of H_{i+1} is smaller. □

Lemma 2.5 *Every definable injective endomorphism f of an ω-stable group G is surjective.*

Proof. $f(G) \subseteq G$ has the same MR and Morley degree. □

Lemma 2.6 *If G is a stable group and $A \subseteq G$ is an abelian subgroup. Then there is a definable abelian subgroup $A' \supseteq A$. If A is nilpotent of class n, then we find A' definable, nilpotent of class n with $A \subseteq A'$.*

Proof. Let $A' = Z_1(C_G(A))$. By Lemma 2.2 A' is definable. □

There are interesting results in algebra:

Theorem 2.7 (Bryant, Hartley) *Every soluble torsion group satisfying the chain condition for centralizers is an extension of a nilpotent normal subgroup by an abelian-by-finite group of finite Prüfer rank. Hence this is true for stable soluble torsion groups.*

Definition Finite Prüfer rank n: Every finitely generated subgroup has a generating set of size at most n.

This means for an abelian torsion group A that it can be embedded into $\bigoplus_{1\leq i\leq n} Z(p_i^\infty)$ for suitable p_i.

Theorem 2.8 (B., Wilson) *Let G be a soluble stable group having a nilpotent normal subgroup N_1 such that G/N_1 is a torsion group. Then G has a nilpotent normal subgroup N such that G/N is an abelian-by-finite torsion group of finite Prüfer rank.*

$\aleph_0$-categorical groups

Theorem 2.9 (U. Felgner (1978); W. Baur, G. Cherlin, A. Macintyre (1979)) *Every $\aleph_0$-categorical stable group G is nilpotent by finite.*

Theorem 2.10 (O. Kegel (1989) uses classification of finite simple groups) *Every locally finite group of bounded exponent, that satisfies the chain condition for centralizers, is nilpotent by finite.*

Theorem 2.11 (W. Baur, G. Cherlin, A. Macintyre) *Every $\aleph_0$-categorical ω-stable group G is abelian by finite.*

Open problem: Is every $\aleph_0$-categorical stable group G abelian by finite?

Background:

A. Lachlan: $\aleph_0$-categorical superstable theories are ω-stable.

E. Hrushovski: There is a stable $\aleph_0$-categorical theory that is not ω-stable.

Connected components

Note:

- If H is a subgroup of G, then the number of left cosets is equal to the number of right cosets

$$(aH)^{-1} = Ha^{-1}.$$

- If H is a subgroup of finite index in G, then there is a normal subgroup H^* of G of finite index such that $H^* \subseteq H \subseteq G$.

Let G be stable and sufficiently saturated. Let $G^0(\varphi(x, \bar{y}))$ be the subgroup of G that is the intersection of all groups of finite index in G defined by $\varphi(x, \bar{a})$ for some parameter $\bar{a}$ in G. By Lemma 2.1 it is a finite intersection and therefore definable. As an intersection of normal subgroups it is normal. It is a characteristic subgroup and therefore $\emptyset$-definable.

$G^0(\varphi)$ φ-connected component;

$G^0 = \bigcap_{\varphi \in L} G^0(\varphi)$ connected component;

G is connected if $G = G^0$.

If G is ω-stable then G^0 is $\emptyset$-definable.

If G is stable G^0 is Λ-definable.

3 Generic types

G stable group not necessarily pure

A definable subset of G.

Essentially due to Poizat based on Cherlin, Zil'ber:

Definition
A is right generic if $G = a_1 A \cup \ldots \cup a_n A$ for some right translates $a_i A$.
A is left generic if $G = A a_1 \cup \ldots \cup A a_n$ for some $a_1, \ldots, a_n$ (iff A^{-1} is right generic).
A is bi-generic if $G = a_1 A b_1 \cup \ldots \cup a_n A b_n$ for some $a_1, \ldots, a_n, b_1, \ldots, b_n$.

We will show that all 3 cases are equivalent. Then we say: A is *generic*.

Definition $p \in S_1(G)$ is generic if it satisfies only generic formulas.
If p is generic, then p^{-1} is generic.

Lemma 3.1 *Let A be a definable subset of a stable group G. Then either A is right generic or its complement $\neg A$ is left generic.*

Proof. Suppose not. Then for any $a_1, \ldots, a_n \in G$ there exists

$$d \notin (\neg A)a_1^{-1} \cup \ldots \cup (\neg A)a_n^{-1}.$$

Hence $da_i \in A$ for $1 \leq i \leq n$. Analogously for any $a_1, \ldots, a_n \in G$ there is some e such that

$$a_i e \in \neg A \quad \text{for } 1 \leq i \leq n.$$

Using this we get by induction $b_1, \ldots, b_i, \ldots, c_1, \ldots, c_i$ such that

$$c_{n+1}b_1, \ldots, c_{n+1}b_n \in A \quad \text{and} \quad c_1 b_{n+1}, \ldots, c_n b_{n+1} \in \neg A.$$

Then if $i < j$ then $c_i b_j \notin A$, if $j < i$ then $c_i b_j \in A$. This gives the order property for the pairs (c_i, b_i).

Corollary 3.2 *There is a bi-generic complete type.*

Proof. We show if $A \cup B$ bi-generic, then either A is generic or B is generic. If $G = a_1(A \cup B)b_1 \cup \ldots \cup a_n(A \cup B)b_n$ then

$$G = (a_1 A b_1 \cup \ldots \cup a_n A b_n) \cup (a_1 B b_1 \cup \ldots \cup a_n B b_n).$$

By Lemma 3.1 the first or the second union is bi-generic. Hence A or B is bi-generic.

We have shown:

A finite union of non-generic definable sets is non-generic and hence $\neq G$.

By compactness we get that negations of non-generic formulas form a consistent set. □

Lemma 3.3 *A generic type does not fork over* $\emptyset$.

Proof. Let $\varphi(x, \bar{y})$ be a formula. We use the local rank $R^{\varphi}_{\omega}(p)$. Let $\psi(\bar{x})$ be with parameters.

i) $R^{\varphi}_{\omega}(\psi(x)) \geq 0$ if $\psi(x)$ is consistent.

If δ is limit ordinal.

ii) $R^{\varphi}_{\omega}(\psi(x)) \geq \delta$ if $R^{\varphi}_{\omega}(\psi(x)) \geq \beta$ for all $\beta < \delta$.

iii) $R^{\varphi}_{\omega}(\psi(x)) \geq \beta + 1$ if for each $i < \omega$ there is Ψ_i which is a finite collection of φ-formulae, such that:

  a) For $i < j < \omega$ Ψ_i and Ψ_j are contradictory (i.e. either there are $\varphi(x, \bar{b}) \in \Psi_i$ and $\neg\varphi(x, \bar{b}) \in \Psi_j$ or there are $\neg\varphi(x, \bar{b}) \in \Psi_i$ and $\varphi(x, \bar{b}) \in \Psi_j$.

  b) For each $i < \omega$ $R^{\varphi}_{\omega}(\psi \wedge \bigwedge \Psi_i) \geq \beta$.

Note:

- If T is stable, then $R^{\varphi}_{\omega}(\psi(x))$ is finite.
- $R^{\varphi}_{\omega}(p) = \min\{R^{\varphi}_{\omega}(\psi(x)) : \psi(\bar{x}) \in p\}$.

Let T be stable. Let $p(x) \in S_n(A)$, $A \subset B$ and let $q(x) \in S_n(B)$ be a forking extension of p. Then there is some $\varphi(x, \bar{y})$ such that $R^{\varphi}_{\omega}(p) > R^{\varphi}_{\omega}(q)$.
Let p be a generic type over M where G is defined in M. The parameters are in $\mathrm{acl}(\emptyset)$. It is sufficient to show, that $R^{\varphi}_{\omega}(p)$ is maximal for all $\varphi(x, \bar{y})$.
It is sufficient to consider formulas $\varphi(u \cdot x \cdot v, \bar{y})$ only. Let q be a type where $R^{\varphi}_{\omega}(q)$ is maximal.
If $\psi(x) \in p$, then $\psi(G)$ is generic. Hence there are a, b with $a\psi(x)b$ is in q. Hence $\psi(x)$ is of maximal R^{φ}_{ω}-rank. □

Lemma 3.4 *For every $\varphi(x, \bar{y})$ there is a natural number $n(\varphi)$ such that if $\varphi(G, \bar{a})$ is a generic set, then G is covered by $n(\varphi)$ sets $a\,\varphi(G, \bar{a})b$.*

Proof. Let $\mathbb{G}$ be highly saturated. Let p be generic over $\mathbb{G}$. If $\varphi(\mathbb{G}, \bar{a})$ is generic, then $\varphi(axb, \bar{a}) \in p$ for some a, b. Hence $\varphi(\mathbb{G}, \bar{a})$ is generic iff $\mathbb{G} \models \exists uv\, d_{\varphi}(u, v, \bar{a})$ where d_{φ} is the definition of p for $\varphi(uxv, \bar{y})$. The result follows from compactness. □

The lemma provides us a formula $\psi_{\varphi}(\bar{y})$ such that: $\varphi(\bar{x}, \bar{a})$ is generic iff $\mathbb{G} \models \psi_{\varphi}(\bar{a})$.
Note generic types are defined over models G. By Lemma 3.3 they do not fork over $\emptyset$. Hence it makes sense to consider their restrictions to subsets of G as generic types.

Lemma 3.5 *a) Nonforking extensions of generic types are generic.*

b) If a and b are generic and independent over A, then ab is generic. Furthermore a and ab and b and ab are independent.

Proof. a) By the definition it is sufficient to consider a generic type p over G and its heirs. Then formulas $\varphi(x, \bar{y}) \wedge \neg\psi_{\varphi}(\bar{y})$ are not in $\mathrm{cl}(p)$ (class of fundamental order). Then this is true for every heir and the heirs are generic.

b) W.l.o.g. $A = G$. Otherwise consider a non-forking extension of $\mathrm{tp}(a, b/A)$ over G. Choose $G_1 \succeq G$, $b \in G_1$, $G_1 \underset{G}{\perp} a$. Then a is generic over G_1. Hence ab is generic over G_1 and therefore over G. (A bi-generic $\Rightarrow$ Ab bi-generic.) Since $\mathrm{tp}(ab/G_1)$ dnf G, $ab \underset{G}{\perp} b$. Analogously $ab \underset{G}{\perp} a$. □

Corollary 3.6 *Every b is the product of two generics. If $\mathrm{tp}(b/G) = p$ is given and $\varphi(x)$ is a bi-generic formula, then there is a $c' \in G$ such that $c'p \vDash \varphi(x)$.*

Proof. Assume $\mathrm{tp}(a/b) \ni \varphi(x)$ is generic and $a \mathop{\smile\!\!\!\!\!\mid}_G b$. Then $\mathrm{tp}(a/Gb)$ is generic (Lemma 3.5). Hence $\mathrm{tp}(ab^{-1}/Gb)$ is generic. $ab^{-1} \mathop{\smile\!\!\!\!\!\mid}_G b$ and $\mathrm{tp}(ab^{-1}/G)$ is generic (Lemma 3.3). If $c = ab^{-1}$, then $c^{-1}a = b$ and $a = cb$. Since $b \mathop{\smile\!\!\!\!\!\mid}_G c$, there is some $c' \in G$ such that $\vDash \varphi(c'b)$. □

Corollary 3.7 *A formula $\varphi(x)$ is right-generic iff it is left-generic iff it is bi-generic.*

Proof. It is sufficient to show that bi-generic formulas $\varphi(\bar{x})$ are right- and left-generic. Corollary 3.6 says that every element of G can be moved into $\varphi(x)$ by left-multiplication. By compactness $\varphi(x)$ is right-generic. The "left-generic" case is proved analogously. □

G^0 and the generics

$$G^0 = \bigcap_{\varphi \in L} G^0(\varphi), \qquad G^0(\varphi) \quad \emptyset\text{-definable.}$$

In a sufficiently saturated model $G^0 \neq \langle 1 \rangle$. We consider the action of G on $S_1(G)$ by $gp = \{\varphi(x) : \varphi(gx) \in p\}$. If $G \preceq \mathbb{G}$ and $\mathrm{tp}(a/G) = p$ for $a \in \mathbb{G}$, then ga realizes gp.

Definition $\mathrm{Stab}(p) = \{g \in G : gp = p\}$.

Consider formulas $\varphi(u \cdot x, \bar{y})$.

Definition

$$\begin{aligned} \mathrm{Stab}^{\varphi}(p) &= \{g \in G : gp \restriction \varphi = p \restriction \varphi\} \\ &= \{g \in G : \varphi(hx, \bar{b}) \in p \text{ iff } \varphi(hgx, \bar{b}) \in p \text{ for all } h; \bar{b} \text{ in } G\} \\ &= \{g \in G : \forall z \forall \bar{y} [dp(\varphi(zx, \bar{y})) \leftrightarrow dp(\varphi(zgx, \bar{y}))]\}. \end{aligned}$$

Hence $\mathrm{Stab}^{\varphi}(p)$ is a definable subgroup.

$$\mathrm{Stab}(p) = \bigcap_{\varphi} \mathrm{Stab}^{\varphi}(p) \quad \text{ is } \quad \wedge\text{-definable.}$$

We work with $\varphi(u \cdot x, \bar{y})$ to ensure that $\mathrm{Stab}^{\varphi}(p)$ is a subgroup!

Lemma 3.8 $\mathrm{Stab}^{\varphi}(p) \subseteq G^0(\varphi)$ *and* $\mathrm{Stab}(p) \subseteq G^0$ *for* $p \in S_1(G)$.

Proof. p contains the information about the coset modulo $G^0(\varphi(ux, \bar{y}))$. Hence $\mathrm{Stab}^{\varphi}(p) \subseteq G^0(\varphi(ux, \bar{y}))$: Let $\psi(x)$ define $G^0(\varphi)$. Ex. $b \in G$ with $\psi(b^{-1}x) \in p$. If $g \in \mathrm{Stab}^{\varphi}(p)$ then $\psi(b^{-1}gx) \in p$. Hence $b^{-1}gb$ and $g \in G^0(\varphi)$. Note $\psi(x)$ defines a normal subgroup. □

Lemma 3.9 *If p is generic, then* $\mathrm{Stab}^{\varphi}(p)$ *has finite index in G and* $\mathrm{Stab}(p) = G^0$*. For every formula φ, the generic types have only finitely many pairwise distinct φ-types.*

Proof. The first assertion follows from the second. (If p generic, then ap generic.) The number of types over $\emptyset$ is bounded. Since the generic types do not fork over $\emptyset$, their number is bounded. Hence there are only finitely many φ-types of generics, since otherwise we could produce to many by compactness. □

Theorem 3.10 *There is a unique generic in every coset of G^0. If G is sufficiently saturated, then G acts transitively on its generics. p is generic iff* $\mathrm{Stab}(p) = G^0$.

Proof. If G is sufficiently saturated, then every coset of G^0 is represented in G. There is a generic type in G^0 (principal generic). By translation in every coset of G^0 there is a generic. We have to show that there are not two generics in G^0.

We choose realizations a and b independent over G. Since $\mathrm{Stab}(\text{generic}) = G^0$, b and ab realize the same type over $G\hat{}a$ and therefore over G. Similarly a and ab have the same type over G. Hence a and b have the same type over G. Hence there is exactly one generic in a coset of G^0 given by a translate of the principal generic.

If p is generic, then $\mathrm{Stab}(p) = G^0$ by Lemma 3.9. Now assume $\mathrm{Stab}(p) = G^0$. Then this is true for every heir of p. Let a realize p and b realize the principal generic over $G\hat{}a$. Then $a \underset{G}{\downarrow} b$. a and ba have the same type over $G\hat{}b$. Furthermore ba is generic over $G\hat{}a$. Therefore a is generic over G. □

Theorem 3.11 *Let K be a stable infinite field. Then K has no definable additive or multiplicative subgroup of finite index. Its additive generic is also the unique multiplicative generic.*

Proof. Assume H has finite index in K^+. Then aH is also such a subgroup. Hence the intersection of all aH is a finite intersection and it is an ideal I.

Since K is infinite, $I \neq 0$ and $I = K$.
Let p be the unique generic of K^+. p is in $K^\times$. $x \to ax$ is an additive automorphism. It preserves the additive generic. Hence $ap = p$ for every $a \in K^\times$. By Theorem 3.10 p is the multiplicative generic. $K^\times$ is connected.□

4 Groups of finite Morley rank

Definition G is minimal, if every proper definable subgroup is finite. E.g. connected group of MR 1.

Lemma 4.1 *If G is ω-stable, then the generic types are the types of maximal MR.*

Proof. $(\rightarrow)$ Every generic set is of maximal Morley rank.

$(\leftarrow)$ Let p be of maximal Morley rank. By Theorem 3.10 it is sufficient to show that $\mathrm{Stab}(p) = G^0$. This is clear since $\{ap : a \in G\}$ if finite. □

Theorem 4.2 (J. Reineke) *Every infinite minimal ω-stable group is abelian.*

Proof. If a is not in the center of G, then $C(a)$ is finite. Let b be generic over a. $\{c : cac^{-1} = bab^{-1}\}$ is finite, since $C(a)$ is finite. Hence b is algebraic over $a\hat{\ } bab^{-1}$. We get $\mathrm{MR}(bab^{-1}/a) = \mathrm{MR}(b/a)$. Therefore the conjugacy class of non-central elements a is generic. By assumption G is connected. There is only one generic type. The generic conjugacy classes of non-central elements a, a' coincide. Hence a and a' are conjugated. If G is not abelian, then $Z(G)$ is finite and $H = G/Z(G)$ is infinite and all elements $\neq 1$ are conjugated. Choose $a \in H$, $a \neq 1$. Then $a^2 \neq 1$, since otherwise $H^2 = 1$ and H abelian. Then there is some b with $b^{-1}ab = a^{-1}$ and therefore $b^{-1}a^{-1}b = a$, because $b^{-1}xb$ is an automorphism. We get $b^{-2}ab^2 = a$ and $C(b) \subsetneq C(b^2)$, $b^2 \neq 1$. Choose c with $c^{-1}bc = b^2$. Finally

$$C(b) \subsetneq C(cbc^{-1}) \subsetneq \ldots \subsetneq C(c^n bc^{-n}) \subsetneq \ldots$$

a contradiction to stability. □

Corollary 4.3 *If* $\mathrm{MR}(G) = 1$*, then G is abelian by finite. Every ω-stable group contains an infinite definable abelian subgroup.*

Proof. If $\mathrm{MR}(G) = 1$ then G^0 is minimal. If $\mathrm{Th}(G)$ is ω-stable choose H infinite definable of minimal MR. Then H^0 is minimal. □

(True for superstability!)

Definition A definable subset $X \subseteq G$ is indecomposable, if whenever H is a definable subgroup of G the coset space X/H is either infinite or contains a unique element.

Example: If X is an infinite connected definable group, then X is indecomposable. H definable subgroup. Then X/H and $X/X \cap H$ have the same number of cosets.

Lemma 4.4 RM(Stab(p)) $\leq$ RM(p) *for* $p \in S_1(G)$.

Proof. Assume $G \preceq G_1$, $a, b \in G_1$, $\mathrm{tp}(a/G) = p$, $b \in \mathrm{Stab}(p)$ with $\mathrm{RM}(b/G) = \mathrm{RM}(\mathrm{Stab}(p))$, $a \underset{G}{\perp} b$. Then

$$\mathrm{RM}(ba/G\hat{\ }a) = \mathrm{RM}(b/G\hat{\ }a) = \mathrm{RM}(b/G) = \mathrm{RM}(\mathrm{Stab}(p))$$

and

$$\mathrm{RM}(ba/G\hat{\ }a) \leq \mathrm{RM}(ba/G) = \mathrm{RM}(p).$$

□

The following results are mainly due to B. Zil'ber.

Theorem 4.5 (Zil'bers Indecomposability Theorem) *Let G be a group of finite Morley rank and $\{X_i : i \in I\}$ a collection of definable indecomposable subsets of G containing 1. Then the subgroup of G generated by $\bigcup_{i \in I} X_i$ is definable and connected.*

Proof. If $\sigma = (i_1, \ldots, i_n) \in I^{<\omega}$, then define $X^\sigma = \{a_1 \ldots a_n : a_1 \in X_{i_1} \ldots a_n \in X_{i_n}\}$. There is some σ such that $\mathrm{RM}(X^\sigma) = k$ is maximal, since $\mathrm{MR}(G)$ is finite.
Let $p \in S_1(G)$ be a type with "$x \in X^\sigma$" and $\mathrm{MR}(p) = k$. We consider $H = \mathrm{Stab}(p)$ and show $X_i \subseteq H$ for each $i \in I$. Otherwise $|X_i/H| > 1$ since $1 \in X_i \cap H$ and $X_i \nsubseteq H$. Since X_i is indecomposable there are $a_1, a_2, \ldots \in X_i$ with $a_i H \neq a_j H$ for $i \neq j$. Then $a_i^{-1} a_j \notin H = \mathrm{Stab}(p)$ implies $a_i p \neq a_j p$. We obtain infinitely many distinct types of MR $k : a_1 p, a_2 p, \ldots$ All these types contain the definable set $X_i X^\sigma$. Hence $X_i X^\sigma$ has MR $= k+1$, a contradiction.

We have shown $\langle \bigcup_{i \in I} X_i \rangle \subseteq H$. We want to show equality. By Lemma 4.4

$$\mathrm{RM}(H) \leq \mathrm{RM}(p) = \mathrm{RM}(X^\sigma) \leq \mathrm{RM}(H).$$

Now $p \in H$ and $\mathrm{RM}(p) = \mathrm{RM}(H)$. Therefore p is a generic type of H. By Theorem 3.10 H acts transitively on its generic types. Since $H = \mathrm{Stab}(p)$

H fixes p and H is connected. Because $X^\sigma \subseteq H$ is generic by Corollary 3.6 $H = X^\sigma \cdot X^\sigma$. Hence H is the group generated by $\bigcup_{i \in I} X_i$ and it is connected.

□

Corollary 4.6 *In Theorem* 4.5 *there are* $X_{i_1}, \ldots, X_{i_m}$ *such that*

$$\left\langle \bigcup_{i \in I} X_i \right\rangle = X_{i_1} \cdot \ldots \cdot X_{i_m} X_{i_1} \cdot \ldots \cdot X_{i_m}.$$

Action of a group H on X is a map

- $\gamma : H \times X \to X$ such that $\gamma(1, x) = x$, $\gamma(h_1, \gamma(h_2, x)) = \gamma(h_1 h_2, x)$.
- $Y \subseteq X$ is H-invariant if $\gamma(Y) = Y$.
- H acts transitively on $Y \subseteq X$, if for all $y_1, y_2 \in Y$ there exist some $h \in H$ with $h(y_1) = y_2$.

Examples:

- $X = A$ is a group and for each h $\ x \to \gamma(h, x)$ is an automorphism of A. We say γ is the action of the group H as a group of automorphisms of A.
- $H = X$ is a group, $\gamma(h, x) = h^{-1} \times h$.
- $H = K^\times$ and $X = A = K^+$. K field, $\gamma(h, x) = hx$.

Definition $\gamma : H \times X \to X$ is an ω-stable action if γ, H and X are defined in a ω-stable structure.

Lemma 4.7 *Assume we have an ω-stable action of a group H on a group G as a group of automorphisms where $X \subseteq G$ is H-invariant. If for all definable H-invariant subgroups J of G either $|X/J| = 1$ or X/J is infinite, then X is indecomposable.*

Proof. Suppose there is some definable subgroup J of G such that $1 < |X/J| < \aleph_0$. Then $X \subseteq a_1 J \cup \ldots \cup a_n J$. If $h \in H$ and $a \in X$, then $\gamma(h^{-1}, a) \in X$. Hence $\gamma(h^{-1}, a) = a_i j$ for some $j \in J$ and $a = \gamma(h, a_i)\gamma(h, j)$. Thus

$$X \subseteq \gamma(h, a_1) \cdot \gamma(h, J) \cup \ldots \cup \gamma(h, a_n)\gamma(h, J)$$

and therefore

(∗) $X/\gamma(h,J)$ is finite for any $h \in H$.

Let $J^* = \bigcap_{h\in H} \gamma(h,J)$. By the Descending Chain Condition there are $h_1,\ldots,h_m$ such that

$$J^* = \bigcap_{1\le i\le m} \gamma(h_i,J).$$

By (∗) X/J^* is finite. Hence $1 < |X/J^*| < \aleph_0$. Since J^* is H-invariant, this is a contradiction. □

Corollary 4.8 i) *Let H be a definable connected subgroup of a ω-stable group G. Then $g^H = \{h^{-1}gh : h \in H\}$ is indecomposable for a fixed $g \in G$.*

ii) *If G is connected ω-stable and of finite Morley rank, then G' is connected and definable.*

Proof. i) $g \to g^h$ is a definable action of H on G. g^H is H-invariant. By Lemma 4.7 it is sufficient to consider definable J in G with $h^{-1}Jh = J$ for all $h \in H$. If g^H/J is finite, then we choose m minimal such that

$$g^H \subseteq a_1J \cup \ldots \cup a_mJ \quad \text{for } a_1,\ldots,a_m \in g^H.$$

Since J is H-invariant and $a_i^h \in g^H$ we have $a_i^hJ = a_jJ$ for some j. We get a transitiv definable action γ of H on $\{a_1,\ldots,a_m\}$. (Transitive since $a_i = g^{h_i}$.) If H^* is the subgroup of all $h^* \in H$ with $\gamma(h^*,a_1) = a_1$, then H^* has finite index in H. Hence $H^* = H$ and $m = 1$.

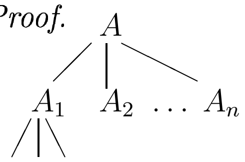
ii) Since g^G is indecomposable also $g^{-1}g^G$ is indecomposable and contains 1. By Zil'ber's Theorem G' is definable and connected. □

Theorem 4.9 *If g is infinite non-abelian of finite Morley rank such that G has no non-trivial definable proper normal subgroup. Then G is simple.*

Proof. G is connected. If a^G is finite, then $C(a)$ has finite index in G. Hence a is central and therefore $a = 1$ since G is non-abelian. Therefore for $a \neq 1$ each $\{1\} \cup a^G$ is infinite and indecomposable (Lemma 4.7). By Theorem 4.5 $\{1\} \cup a^G$ generates a definable normal subgroup of G and therefore G. Since this is true for every a G is simple. □

Lemma 4.10 *Let A be a definable subset of G where G is a ω-stable group. Then A is the union of finitely many maximal indecomposable subsets, which are pairwise disjoint.*

Proof.



$$A = \cup A_i$$

$$A_i \subseteq aH$$

finitely branching tree

No infinite branch since there is no descending chain of subgroups.
Königs Lemma: Finite Tree.
Hence $A = \bigcup_{1 \le i \le \kappa} B_i \quad B_i$ indecomposable pairwise disjoint.
If B and C are definable indecomposable subsets and $B \cap C \neq \emptyset$, then $B \cup C$ indecomposable. □

Theorem 4.11 *Simple groups and fields of finite Morley rank are $\aleph_1$-categorical.*

Proof. G simple group, A infinite definable subset. Let $A_1 \subseteq A$ be a definable maximal indecomposable subset of A (Lemma 4.10). For $a \in A_1 \quad B = a^{-1}A_1$ is indecomposable and $1 \in B$, $b^{-1}Bb$ is indecomposable and $1 \in b^{-1}Bb$ for all $b \in B$.
Hence by Theorem 4.5 (Corollary 4.6) G is a product of finitely many $b^{-1}Bb$.
Hence for every definable A there are $b_1 \dots b_n$ in G such that

$$G = (a^{-1}A)^{b_1} \dots (a^{-1}A)^{b_n}.$$

It follows that G has no Vaughtian pairs. If K is a field, then we do the same in the additive group. Choose $A \supseteq A_1 \ni a$ as above and $B = A_1 - a$. Then all bB are indecomposable and contain 0. By Corollary 4.6 for every A there are $b_1, \dots, b_n$ such that

$$K = b_1(A - a) + \dots + b_n(A - a).$$

Again we have no Vaughtian pairs and therefore $\aleph_1$-categoricity. □

Definition The action $\gamma : H \times A \to A$ is faithful if for every pair $h_1, h_2 \in H$ there is some $a \in A$ with $\gamma(h_1, a) \neq \gamma(h_2, a)$.

Theorem 4.12 *Let $(H, \cdot)$ and $(A, +)$ be infinite abelian groups with a faithful ω-stable action of H on A as a group of automorphisms. Assume that there is no infinite definable subgroup B of A that is H-invariant. Then we can interpret an algebraically closed field.*

Proof. A^0 is invariant under all definable automorphisms. Hence it is H-invariant and $A = A^0$. Let $a \in A$ be generic over all used parameters.

Claim 1. Ha is infinite.
If Ha is finite, then $H^0 a$ is finite and therefore $H^0 a = \{a\}$, as in the proof of Corollary 4.8.
Let X be $\{b \in A : H^0 b = \{b\}\}$. Then X is generic and every element of A is a product of two elements of X. Then $H^0 b = \{b\}$ for all $b \in A$. Because H acts faithfully $H^0 = \{1\}$ and H is finite, a contradiction.

Claim 2. $Ha \cup \{0\}$ is indecomposable, $Ha \cup \{0\}$ is H-invariant.
By Lemma 4.7 we consider only H-invariant subgroups B of A. By assumption they are finite. Therefore $Ha \cup \{0\}/B$ is infinite (Claim 1) as desired.

By Theorem 4.5 $\langle Ha \cup \{0\} \rangle$ is a definable infinite H-invariant subgroup of A. Hence it is A. There is some n such that each element of A is the sum of n elements in $Ha \cup \{0\}$.
Let $\mathrm{End}(A)$ be the ring of endomorphisms of A. Let R be the subring generated by H in $\mathrm{End}(A)$. R is commutative. If $b \in A$, then $b = \sum_{1 \le i \le m} h_i a$ for some $h_i \in H$ and $m \le n$.
For $r \in R$ we have

$$r(b) = \sum_{1 \le i \le m} r h_i a = \sum_{1 \le i \le m} h_i r a$$

by commutativity of R. If $r_1, r_2 \in R$ and $r_1 a = r_2 a$, then $r_1 = r_2$. Let $ra = b$ and $b = \sum_{1 \le i \le m} h_i a$. Then $h_1 + \ldots + h_m \in R$ and $r = (h_1 + \ldots + h_m)$. Hence for every $r \in R$ there are $h_1, \ldots, h_n$ from $H \cup \{0\}$ such that $r = h_1 + \ldots + h_n$. (We define $0a = 0$.)

Claim 3. The ring R is interpretable. We define

$$(h_1, \ldots, h_n) \sim (g_1, \ldots, g_n) \quad \text{if and only if} \quad \sum h_i a = \sum g_i a,$$
$$\bar{h} \oplus \bar{g} = \bar{\ell} \quad \text{if and only if} \quad \sum h_i a + \sum g_i a = \sum \ell_i a$$

and

$$\bar{h} \otimes \bar{g} = \bar{\ell} \quad \text{if and only if} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} h_i g_j a = \sum_{k=1}^{n} \ell_k a.$$

Claim 4. R is a field.
Let $r \in R$, $r \neq 0$. Let $b \in B = \ker(r)$. Then $rb = 0$ and for any $h \in H$

$$r(hb) = (rh)b = (hr)b = h(rb) = 0.$$

Thus B is H-invariant and by assumption B is finite. Because A is connected r is surjective. Choose $c \in A$ with $rc = a$. Let $c = \sum h_i a$ and $s = \sum h_i \in R$. Then $sa = c$ and $rsa = a$. $1a = a$ implies $rs \approx 1$. By Macintyre R is algebraically closed. □

Theorem 4.13 (Zil'ber) *If G is an infinite connected, solvable, non-nilpotent group of finite Morley rank, then G interprets an algebraically closed field.*

5 One-based groups

Definition A stable theory T is one-based if for every n every model $M \models T$, every $p \in S_n(M)$, and every realization $\bar{a}$ of p we have $\mathrm{Cb}(p) \in \mathrm{acl}(\bar{a})$.

Theorem 5.1 (E. Hrushovski, A. Pillay) *Let G be a stable one-based group.*

i) *For any n every definable subset X of G^n is a Boolean combination of cosets of* $\mathrm{acl}(\emptyset)$*-definable subgroups of G^n.*

ii) *G is abelian by finite.*

Proof: We assume ω-stability.

Claim (1) Let H be a connected definable subgroup of G. Then $\mathrm{Cb}(H)$ is algebraic over $\emptyset$.

Proof of (1): Assume $G \preceq \mathbb{G}$, $g \in G$, $\mathrm{tp}(g/G)$ is the generic type of G. Let p be the generic of H, p' be the heir of p over $\mathbb{G}$, and $\mathrm{tp}(a/\mathbb{G}) = p'$.
Let $q = \mathrm{tp}(ga/\mathbb{G})$. Let $u = \mathrm{Cb}(H)$, $v = \mathrm{Cb}(q)$. By one-basedness $\nu \in \mathrm{acl}(g \cdot a)$ $u \in G$ and $\mathrm{tp}(ga/G)$ is a generic type of G. Hence $ga \underset{\emptyset}{\perp} u$ (Lemma 3.3).

Furthermore $\mathrm{MR}(q) = \mathrm{MR}(a/\mathbb{G}) = \mathrm{MR}(H)$. To finish the proof we show that u is definable over v.
Then $u \underset{\emptyset}{\perp} ga$ and $v \in \mathrm{acl}(g \cdot a)$ implies $u \in \mathrm{acl}(g \cdot a)$ and $u \in \mathrm{acl}(\emptyset)$.
We consider $f \in \mathrm{Aut}(\mathbb{G})$ with $f(q) = q$. It is sufficient to show that $f(H(G)) = H(G)$.
Let $H_1 = f(H)$ and $g_1 = f(g)$. Then $gH, g_1H_1 \in q$. Hence

$$\mathrm{MR}(q) = \mathrm{MR}(H) \geq \mathrm{MR}(gH \cap g_1H_1) \geq \mathrm{MR}(q).$$

We have $\mathrm{MR}(gH \cap g_1H_1) = \mathrm{MR}(H)$. Since $gH \cap g_1H_1 = g_2(H \cap H_1)$ for any $g_2 \in gH \cap g_1H_1$, we have $\mathrm{MR}(H \cap H_1) = \mathrm{MR}(H)$. Since H is connected $H_1 = H$ and $gH = g_1H$. □ Proof of (1)

Proof of (ii): We have to show G^0 is abelian. Hence w.l.o.g. $G = G^0$. We apply (1) to the group G^2: Let H_g be the subgroup $\{(h, g^{-1}hg) : h \in G\}$ of G^2. We define an equivalence relation $g \sim g'$ iff $H_g = H_{g'}$. H_g is definably isomorphic with G. Hence it is connected. Furthermore $g \sim g'$ iff for all $h \in G$

$$g^{-1}hg = g'^{-1}hg \text{ iff } g = g' \bmod Z(G).$$

Hence $\sim$ is definable. By (1) each H_g is $\mathrm{acl}(\emptyset)$ definable. Then there are at most countably many H_g and $\sim$ has at most countably many classes. By compactness there are only finitely many $\sim$ classes and $Z(G)$ has finite index in G. □ Proof of (ii)

Claim (2) For every $n \in \omega$ every $p \in S_n(G)$ there exists $b \in G$ such that

$$\mathrm{Stab}(p)b \in p.$$

Proof of (2): W.l.o.g. $n = 1$ and $\mathrm{Cb}(p) = \emptyset$ (we extend the language if necessary). As above $G \preceq \mathbb{G}$, $g \in \mathbb{G}$, $\mathrm{tp}(g/G)$ is the generic type of G, $p' = \mathrm{tp}(a/\mathbb{G})$ is the heir of p and $q = \mathrm{tp}(ga/\mathbb{G})$. Since g generic and $g \underset{G}{\perp} a$, we have g is generic over $G\hat{\ }a$ and therefore

(3) $\quad a \underset{G}{\perp} ga.$

Let u be $\mathrm{Cb}(g\,\mathrm{Stab}(p))$ and v be $\mathrm{Cb}(g)$. We show that

(4) $\quad u$ and v are interdefinable.

Let $f \in \mathrm{Aut}(\mathbb{G})$. We show

$$f(g\,\mathrm{Stab}(p)) = g\,\mathrm{Stab}(p) \text{ iff } f(q) = q.$$

Since $q = gp'$ and $\mathrm{Cb}(p') = \emptyset$, we have $f(q) = f(g) \cdot f(p') = f(g)p'$. Hence

$$\begin{aligned} f(q) = q \quad & \text{iff } gp' = f(g)p' \\ & \text{iff } g^{-1}f(g) \in \mathrm{Stab}(p) \\ & \text{iff } g\,\mathrm{Stab}(p) = f(g)\,\mathrm{Stab}(p) \\ & \text{iff } g\,\mathrm{Stab}(p) = f(g\,\mathrm{Stab}(p)), \end{aligned}$$

since $\mathrm{Cb}(\mathrm{Stab}(p)) = \emptyset$. □ Proof of (4)

Since G is one-based, $v \in \mathrm{acl}(ga)$ and $u \in \mathrm{acl}(ga)$ by (4). Then $a \underset{G}{\perp} ga \cup \{u\}$ by (3).

Assume $a' \in \mathbb{G}$ with $\text{tp}(a'/G) = p$ and $a' \underset{G}{\perp} u$.

Since $v = \text{Cb}(q)$ is definable over u (4), $\mathbb{G} \underset{G\cup\{u\}}{\perp} ga$.

Hence $a' \underset{G\cup\{u\}}{\perp} ga$ and $a' \underset{G}{\perp} ga \cup \{u\}$.

We get

(5) $\quad \text{tp}(ga, u, a/G) = \text{tp}(ga, u, a'/G)$.

Using the parameter u there is a formula in the left type saying $ga \in (g\,\text{Stab}(p)) \cdot a$. Hence we have $ga \in (g\,\text{Stab}(p)) \cdot a'$ and $a \in \text{Stab}(p) \cdot a'$. Hence for any two independent (over G) realizations a and a' of p we have $a \in \text{Stab}(p)a'$.

Hence there is some $b \in G$ such that

$$x \in \text{Stab}(p) \cdot b \in p.$$

(The formula is presented in $\text{cl}(p)$.) $\square$ Proof of (2)

To prove (i) we show

Claim (6) Assume $p, p' \in S_n(G)$ and for any definable subgroup H of G and any $a \in G$ we have

$$Ha \in p \ \text{ if and only if } \ Ha \in p'.$$

Then $p = p'$.

Proof of (6): W.l.o.g. $n = 1$, $\text{MR}(p') \geq \text{MR}(p)$, $c \models p$, $c' \models p'$, $c \underset{G}{\perp} c'$.

Then by (2) $c \in \text{Stab}(p)a$ for some $a \in G$. Hence by assumption $c' \in \text{Stab}(p)a$ and therefore $c'c^{-1} \in \text{Stab}(p)$. We show $c'c^{-1} \underset{G}{\perp} c$:

$$\begin{aligned} \text{MR}(c'c^{-1}/G \cup \{c\}) &= \text{MR}(c'/G \cup \{c\}) \\ &= \text{MR}(p') \\ &\geq \text{MR}(p) \\ &\geq \text{MR}(\text{Stab}(p)) \quad \text{Lemma 4.4} \\ &\geq \text{MR}(c'c^{-1}/G), \quad \text{since } \ c'c^{-1} \in \text{Stab}(p). \end{aligned}$$

Choose $G' \succeq G$, $c'c^{-1} \in G'$ and $G' \underset{G}{\perp} c$. Then

$$\text{tp}(c'/G') = \text{tp}(c'c^{-1}c/G') = c'c^{-1}\text{tp}(c/G') = \text{tp}(c/G').$$

Hence $\text{tp}(c'/G')$ is a non-forking extension of p and therefore $p = p'$ since $\text{tp}(c'/G) = p'$. $\square$

6 Some further topics

– Group Configurations

– Borovik-Program

– Model Theory of Free Groups