Generic variations of models of T

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Abstract

Let T be a model-complete theory that eliminates the quantifier $\exists^\infty x$. For T we construct a theory T^+ such that any element in a model of T^+ determines a model of T. We show that T^+ has a model companion T^1. We can iterate the construction. The produced theories are investigated.

1 Introduction

In [CP] Zoe Chatzidakis and Anand Pillay construct generic structures to obtain simple theories. We combine their point of view with an idea in [B] and construct model companions of a new type of theories. But only in few cases we obtain simple theories. Following a suggestion of Lou van den Dries we also use a paper of Peter Winkler [W]. He considers model-complete theories T that eliminate the quantifier $\exists^\infty x$. He shows that certain expansions of T, in particular every Skolem expansion, have a model-completion.

In this paper we also start with a model-complete theory T that eliminates the quantifier $\exists^\infty x$. Let L be the language of T. We replace each n-ary non-logical symbol $R(x_1, \dots, x_n)$, $f(x_1, \dots, x_n)$, or c (if $n = 0$) by a $(n+1)$-ary symbol $R^+(x_0, x_1, \dots, x_n)$, $f^+(x_0, x_1, \dots, x_n)$, or $c^+(x_0)$ respectively. Let L^+ be the new language. Let T^+ be the theory of all L^+-structures M with the following property: If we fix any element a in M, then the relations $R^{+M}(a, x_1, \dots, x_n)$, the functions $f^{+M}(a, x_1, \dots, x_n)$, and the constants $c^{+M}(a)$ determine a model of T on $\mathrm{dom}(M)$. The main result is that T^+ has a model companion T^1 and T^1 again eliminates the quantifier $\exists^\infty x$. Hence we can iterate the construction. We define $T^{n+1} = (T^n)^1$. This is the content of chapter 2. Before we continue we will give two examples:

1. Let T be the theory of an infinite and coinfinite unary predicate. Then for $n \geq 1$ T^n is the theory of a random $(n+1)$-ary predicate. In this case the T^n are simple theories. They are not stable. Note that the existence of an infinite and coinfinite unary definable predicate in a model of T implies the independence property for T^1.

2. Let T be the theory of an equivalence relation with infinitely many classes and each class infinite. In this case the T^n $(n \geq 1)$ are not simple, as in most of the

cases. This example was developed in [B] to investigate the relative strength of the Magidor-Malitz quantifiers Q_1^n where $M \models Q_1^n x_1 \ldots x_n \varphi(x_1, \ldots, x_n)$, if there is an uncountable subset $X \subseteq M$ such that $M \models \varphi(a_1, \ldots, a_n)$ for all pairwise distinct $a_1, \ldots, a_n$ in X. Under the assumption of $\diamondsuit_{\omega_1}$ in [B] for every $n \geq 1$ models M_0 and M_1 of T^n are constructed such that M_0 and M_1 are equivalent with respect to the logic with the quantifier Q_1^{n+1} but

$$M_0 \models Q_1^{n+2} x_1 \ldots x_{n+2} R(x_1, \ldots, x_{n+2})$$

and

$$M_1 \models \neg Q_1^{n+2} x_1 \ldots x_{n+2} R(x_1, \ldots, x_{n+2})$$

where R is the relation of the language of T^n.

In chapter 3 at first we show that the constructed theories T^1 are complete. Then we formulate the notion of completeness over algebraically closed sets. It is weaker than elimination of quantifiers. We show that this property is preserved under our construction. In this case we get that the algebraical closure of T^1 is the "transitive closure of the algebraic closure of T".

In chapter 4 simplicity, independence property and strict order property are considered. Especially we assume that T has quantifier elimination and $\mathrm{acl}(A) = A$ for all $A \subseteq M \models T$. In this case elimination of $\exists^\infty x$ is implied. Then all T^n have elimination of quantifiers and $\mathrm{acl}(A) = A$ for all $A \subseteq M \models T^n$. For these T the following property is preserved: $\mathrm{tp}(C/B)$ does not fork over A if and only if $C \cap (B \cup A) = C \cap A$. For stable T with elimination of quantifiers and $\mathrm{acl}(A) = A$ for all $A \subseteq M \models T$ we show that T^1 has not the strict order property.

2 The construction

Let T be a countable theory without finite models. Let L be the corresponding language.

Definition The quantifier $\exists^\infty x_1 \ldots x_n$ (short $\exists^\infty \bar{x}$) is defined in the following way. For all L-structures M, all tuples $\bar{b}$ in M and all formulas $\varphi(\bar{x}, \bar{y})$ in L we have $M \models \exists^\infty x_1 \ldots x_n \varphi(x_1, \ldots, x_n, \bar{b})$, if there are infinitely many pairwise disjoint n-tuples $\bar{a}^i = (a_1^i, \ldots, a_n^i)$ in M such that $M \models \varphi(a_1^i, \ldots, a_n^i, \bar{b})$ for all i.

Note that $M \models \exists^\infty x_1 \ldots x_n \varphi(x_1, \ldots, x_n, \bar{b})$ if and only if there are a proper elementary extension N of M and $a_1, \ldots, a_n \in N \setminus M$ with $N \models \varphi(a_1, \ldots, a_n, \bar{b})$. If M is ω_1-saturated, then this is equivalent to: There are $a_1, \ldots, a_n \in M \setminus \mathrm{acl}(\bar{b})$ with $M \models \varphi(a_1, \ldots, a_n, \bar{b})$. This quantifier was introduced by Peter Winkler [W]. $\exists^\infty x_1$ is the well-known quantifier "there exist infinitely many".

Definition T eliminates the quantifier $\exists^\infty x_1 \dots x_n$, if for every formula $\varphi(x_1, \dots, x_n, \bar{y})$ there is a formula $\psi_\varphi(\bar{y})$ such that for every model M of T and $\bar{b}$ in M we have $M \models \exists^\infty x_1 \dots x_n \varphi(x_1, \dots, x_n, \bar{b})$ if and only if $M \models \psi_\varphi(\bar{b})$.

If T eliminates $\exists^\infty x_1 \dots x_n$ for the formula $\varphi(x_1, \dots, x_n, \bar{y})$, then there is a natural number k such that the following formula can play the role of $\psi_\varphi(\bar{y})$:

$$\exists x_1^1 \dots x_n^1 \dots x_1^k \dots x_n^k \Big(\bigwedge_{1 \le \ell, h \le n} \bigwedge_{\substack{1 \le i,j \le k \\ i \ne j}} x_\ell^i \ne x_h^j \wedge \bigwedge_{1 \le i \le k} \varphi(x_1^i, \dots, x_n^i, \bar{y}) \Big).$$

Peter Winkler called $\varphi(x_1, \dots, x_n, \bar{y})$ algebraically bounded in $x_1, \dots, x_n$ with respect to T, if T eliminates $\exists^\infty x_1 \dots x_n$ for $\varphi(x_1, \dots, x_n, \bar{y})$. Furthermore he said T is algebraically bounded, if T eliminates $\exists^\infty x_1$.
The reason for this definition is the following result of his, that we will use later.

Lemma 2.1 *If T eliminates the quantifier $\exists^\infty x_1$, then T eliminates the quantifiers $\exists^\infty x_1 \dots x_n$ for all n.*

We also need the next lemma from [W]:

Lemma 2.2 *If T eliminates the quantifiers $\exists^\infty x_1 \dots x_n$ for all quantifier-free formulas of L, then T eliminates the quantifiers $\exists^\infty x_1 \dots x_n$ for all existential formulas of L.*

Now we start the construction. We choose a new language L^+ that consists of the following non-logical symbols (the equality symbol is considered as a logical one):

- For every n-ary relation symbol $R(x_1, \dots, x_n)$ in L there is an $(n+1)$-ary relation symbol $R^+(x_0, x_1, \dots, x_n)$ in L^+.
- For every n-ary function symbol $f(x_1, \dots, x_n)$ in L there is an $(n+1)$-ary function symbol $f^+(x_0, x_1, \dots, x_n)$ in L^+.
- For every constant symbol c in L there is an unary function symbol $c^+(x_0)$ in L^+.

If M is a L^+-structure and a is an element of M, then let M^a be the L-structure with the same domain that we obtain if we define:

$$\begin{aligned} M^a &\models R(a_1, \dots, a_n), \text{ if } M \models R^+(a, a_1, \dots, a_n), \\ M^a &\models f(a_1, \dots, a_n) = b, \text{ if } M \models f^+(a, a_1, \dots, a_n) = b, \\ M^a &\models c = b, \text{ if } M \models c^+(a) = b \end{aligned}$$

for all $a_1, \dots, a_n, b$ in M.
If $\varphi(x_1, \dots, x_n)$ is a L-formula where the variables with free occurrence are a subset of $\{x_1, \dots, x_n\}$ and x_0 does not occur bounded in $\varphi(x_1, \dots, x_n)$, then let $\varphi^+(x_0, x_1, \dots, x_n)$ be the L^+-formula that we obtain by simultaneous replacement of $R(y_1, \dots, y_m)$ by

$R^+(x_0, y_1, \ldots, y_m)$, of $f(y_1, \ldots, y_m) = y$ by $f^+(x_0, y_1, \ldots, y_m) = y$, and of c by $c^+(x_0)$. Then we have $M \vDash \varphi^+(a, \bar{b})$ if and only if $M^a \vDash \varphi(\bar{b})$.

We define $T^+ = \{\forall x_0\, \varphi^+(x_0) : \varphi \in T\}$. Then we have: A L^+-structure M is a model of T^+ if and only if for every $a \in M$ the L-structure M^a is a model of T.

Now we will define a L^+-theory T^1. If T is model-complete and eliminates the quantifier $\exists^\infty x_1$, then we will show that T^1 is the model-companion of T^+ and that T^1 eliminates again the quantifier $\exists^\infty x_1$.

Hence we can iterate the construction and obtain model-complete theories $T^{i+1} = (T^i)^1$ that eliminate the quantifier $\exists^\infty x_1$.

We assume that T eliminates the quantifier $\exists^\infty x_1$. For $\varphi(\bar{x}, \bar{y})$ in L we denote by $\psi_\varphi(\bar{y})$ the formula given by Lemma 2.1.

Definition A suitable tuple is a sequence $(n, n+m, \varphi_0(\bar{x}, \bar{y}), \ldots, \varphi_{n+m-1}(\bar{x}, \bar{y}))$ such that

- the $\varphi_i(\bar{x}, \bar{y})$ for $i < n+m$ are conjunctions of atomic formulas and negated atomic formulas in L, where

$$\bar{x} = (x_0, \ldots, x_{m-1}) \text{ and } \bar{y} = (y_0, \ldots, y_{n-1}).$$

- for $i < n+m$

$$T \vDash \varphi_i(\bar{x}, \bar{y}) \longrightarrow \bigwedge_{\substack{r \neq s \\ r,s<m}} x_r \neq x_s \wedge \bigwedge_{\substack{r \neq s \\ r,s<n}} y_r \neq y_s \wedge \bigwedge_{\substack{r<n \\ s<m}} y_r \neq x_s.$$

- for $i < m$

$$T \cup \exists \bar{x} \exists \bar{y}\, \varphi_{n+i}(\bar{x}, \bar{y}) \quad \text{is consistent.}$$

Then we define

$$\begin{aligned} T^1 \;=\; T^+ \cup \Big\{ & \forall\, y_0 \ldots y_{n-1} \Big(\bigwedge_{i<n} (\psi_{\varphi_i})^+(y_i, \bar{y}) \longrightarrow \\ & \exists\, x_0 \ldots x_{m-1} \Big(\bigwedge_{i<n} \varphi_i^+(y_i, \bar{x}, \bar{y}) \wedge \bigwedge_{i<m} \varphi_{n+i}^+(x_i, \bar{x}, \bar{y}) \Big) \Big) : \\ & (n, n+m, \varphi_0(\bar{x}, \bar{y}), \ldots, \varphi_{n+m-1}(\bar{x}, \bar{y})) \text{ is a suitable tuple} \Big\}. \end{aligned}$$

It is obvious that T^+ is consistent. To show the consistency of T^1 we prove the following stronger result.

Lemma 2.3 *Assume that T eliminates the quantifier $\exists^\infty x_1$. Let M be a model of T^+. Then it is possible to embed M into a T^1-model N such that for every $a \in M$ $M^a \underset{L}{\preceq} N^a$.*

Proof. A problem over M consists of a suitable tuple

$$\Phi = (n, n+m, \varphi_0(\bar{x},\bar{y}), \ldots, \varphi_{n+m-1}(\bar{x},\bar{y}))$$

and a n-tuple $\bar{b} = (b_0, \ldots, b_{n-1})$ in M such that $M \vDash \psi^+_{\varphi_i}(b_i, \bar{b})$ for $i < n$. A solution of the problem $(\Phi, \bar{b})$ is a m-tuple $\bar{a}$ in a L^+-extension K of M such that

$$K \vDash \bigwedge_{i<n} \varphi^+_i(b_i, \bar{a}, \bar{b}) \wedge \bigwedge_{i<m} \varphi^+_{n+i}(a_i, \bar{a}, \bar{b}).$$

We choose K in such a way that $M^c \underset{L}{\preceq} K^c$ for all $c \in M$. Therefore $(\Phi, \bar{b})$ is also a problem over K. Conversely a problem $(\Phi', \bar{b}')$ over K with $\bar{b}'$ in M is a problem over M. To obtain K we choose a set $\mathrm{dom}(K)$ that contains $\mathrm{dom}(M)$ such that $|\mathrm{dom}(K)| = |\mathrm{dom}(M)| = |\mathrm{dom}(K) \setminus \mathrm{dom}(M)|$. For every $a \in K$ we can put a L-structure on $\mathrm{dom}(K)$ if we fix the realizations of all $R^+(a, x_1, \ldots, x_n)$, $f^+(a, x_1, \ldots, x_n)$ and $c^+(a)$. That means we build K^a. We do this in such a way that every K^a is a T-model and for every $a \in M$ the structure of M^a is not changed and $M^a \preceq K^a$. Since $(\Phi, \bar{b})$ is a problem over M we have $M^{b_i} \vDash \psi_{\varphi_i}(\bar{b})$. By the remark preceding Lemma 2.1 we can choose K^{b_i} and a m-tuple $\bar{a}^i$ in $\mathrm{dom}(K) \setminus \mathrm{dom}(M)$ such that

$$K^{b_i} \vDash \varphi_i(\bar{a}^i, \bar{b}).$$

Now let $\bar{c}$ be any m-tuple in $\mathrm{dom}(K) \setminus \mathrm{dom}(M)$. For $i < m$, since $T \cup \exists\, \bar{x}\bar{y}\, \varphi_{n+i}(\bar{x}, \bar{y})$ is consistent, we can choose K^{c_i} and $\bar{a}^{n+i}$ such that $K^{c_i} \vDash \varphi_{n+i}(\bar{a}^{n+i}, \bar{b})$. We can reorganize K^{b_i} for $0 \leq i < n$ and K^{c_i} for $0 \leq i < m$ in such a way that

$$K^{b_i} \vDash \varphi_i(\bar{c}, \bar{b}) \text{ and } K^{c_i} \vDash \varphi_{n+i}(\bar{c}, \bar{b}).$$

In K^{b_i}, $\bar{c}$ plays the role of $\bar{a}^i$ and in K^{c_i} the role of $\bar{a}^{n+i}$.
Let κ be $|M|$. Let $\{(\Phi_i, \bar{b}_i) : i < \kappa\}$ be an enumeration of the problems over M. (By assumption L was countable.) We choose a chain of T^+-models $M_0 = M \subseteq M_1 \subseteq \ldots \subseteq M_i \subseteq \ldots$ $(i < \kappa)$ such that $M_i^c \preceq M_{i+1}^c$ for $c \in M_i$, $M_\gamma = \bigcup_{i<\gamma} M_i$ for limit ordinals γ, and M_{i+1} solves the problem $(\Phi_i, \bar{b}_i)$. Note that by the remark above $(\Phi_i, \bar{b}_i)$ is also a problem over M_i.
Let M^1 be $\bigcup_{i<\kappa} M_i$. Then M^1 is a T^+-model. Every problem $(\Phi, \bar{b})$ over M is a problem over every M_i $(i < \kappa)$. If $(\Phi, \bar{b}) = (\Phi_i, \bar{b}_i)$, then it has a solution in M_{i+1} and hence in M^1. Now we construct a chain $M \subseteq M^1 \subseteq M^2 \ldots$ of T^+-models such that M^{i+1} solves the problems over M^i and $(M^i)^c \preceq (M^{i+1})^c$ for $c \in M^i$. $N = \bigcup_{i<n} M^i$ is the desired T^1-model. By construction N is a T^+-model. To check the additional axioms we consider a problem $(\Phi, \bar{b})$ over N. Then $\bar{b}$ is in M^i for some $i < \omega$. Hence $(\Phi, \bar{b})$ is a problem over M^i. By construction it has a solution in M^{i+1} and therefore in N. $\square$

Corollary 2.4 *T^1 is consistent.*

Lemma 2.5 *Let $\psi(\bar{z})$ be a quantifier-free L^+-formula. Then $\psi(\bar{z})$ is equivalent to a disjunction of formulas $\exists \bar{x}\, \varphi(\bar{x}, \bar{y}) \wedge \chi(\bar{z})$ with respect to T^+ such that:*

- *The variables in $\bar{y}$ are a subset of the variables in $\bar{z}$.*

- *$\chi(\bar{z})$ is a conjunction of equations $y = z$ where $y \in \bar{y}$, $z \in \bar{z} \setminus \bar{y}$. For each $z \in \bar{z} \setminus \bar{y}$ there is exactly one equation.*

- *$\varphi(\bar{x}, \bar{y}) = \bigwedge_{i<n} \varphi_i^+(y_i, \bar{x}, \bar{y}) \wedge \bigwedge_{j<m} \varphi_{n+j}^+(x_j, \bar{x}, \bar{y})$ where $(n, n+m, \varphi_0(\bar{x}, \bar{y}), \ldots, \varphi_{n+m-1}(\bar{x}, \bar{y}))$ is a suitable tuple.*

- *For every x_i there is an equation $x_i = t$ in $\varphi(\bar{x}, \bar{y})$ where t is a L^+-term over $\bar{y}$ and $\{x_0, \ldots, x_{i-1}\}$.*

Proof. An atomic formula is unnested if it is of the form $x = x$, $x = y$, $x = c$, $x = f(x_1, \ldots, x_\ell)$, or $R(x_1, \ldots, x_\ell)$. Unnested formulas are formulas that contain only unnested atomic formulas (see [H]).
Now we consider the quantifier-free L^+-formula $\psi(\bar{z})$. Let Y be the set of terms in $\psi(\bar{z})$ that are not a variable. (Note that in L^+ there are no constants.) For every $t \in Y$ we introduce a new variable x_t. Let $\bar{x}_Y$ be $(x_t : t \in Y)$. It is well known that there is a quantifier-free unnested formula $\psi^*(\bar{x}_Y, \bar{z})$ such that

$$\models \psi(\bar{z}) \longleftrightarrow \exists \bar{x}_Y \psi^*(\bar{x}_Y, \bar{z}),$$
$$\models \psi^*(\bar{x}_Y, \bar{z}) \longrightarrow x_t = t \quad \text{for all } \ t \in Y.$$

Now we write $\psi^*(\bar{x}_Y, \bar{z})$ in disjunctive normal form. Furthermore we can assume that every conjunction $\varphi^*(\bar{x}_Y, \bar{z})$ in $\psi^*(\bar{x}_Y, \bar{z})$ contains either an equation or an inequation for every pair of distinct variables and all equations $x_t = t$ for the x_t and t in $\varphi^*(\bar{x}_Y, \bar{z})$. Then we reduce the number of variables. If we have an equation between variables, then we replace one of them by the other and cancel the equation. In the case $x_t = z_j$ we use z_j and replace x_t. Denote the result by $\varphi(\bar{x}, \bar{y})$ where $\bar{x}$ is a subsequence of $\bar{x}_Y$ and $\bar{y}$ of $\bar{z}$. We choose a conjunction $\chi(\bar{z})$ of the cancelled equations $z_i = y_j$ with $z_i \in \bar{z} \setminus \bar{y}$, $y_j \in \bar{y}$. W.l.o.g. there is only one for each $z_i \in \bar{z} \setminus \bar{y}$. Then $\psi(\bar{z})$ is logically equivalent to the disjunction of the $\exists \bar{x}\, \varphi(\bar{x}, \bar{y}) \wedge \chi(\bar{z})$. Modulo T^+ we can omit the $\varphi(\bar{x}, \bar{y}) \wedge \chi(\bar{z})$ with $T^+ \models \neg \exists \bar{y} \exists \bar{x}\, \varphi(\bar{x}, \bar{y})$. □

Theorem 2.6 *Assume that T is model-complete and eliminates the quantifier $\exists^\infty x_1$. then T^1 is the model companion of T^+.*

Proof. We have $T^+ \subseteq T^1$. By Lemma 2.3 we obtain $T^+_\forall = T^1_\forall$. It remains to show that T^1 is model-complete.
Let $M \subseteq N$ be two T^1-models. Let $\varphi(\bar{x}, \bar{y})$ be a quantifier-free L^+-formula. Assume $N \models \exists \bar{x}\, \varphi(\bar{x}, \bar{b})$ for some $\bar{b}$ in M. We have to show $M \models \exists \bar{x}\, \varphi(\bar{x}, \bar{b})$. By Lemma 2.5 we can w.l.o.g. assume that $\varphi(\bar{x}, \bar{y})$ has the form

$$\varphi(\bar{x}, \bar{y}) \equiv \bigwedge_{i<n} \varphi_i^+(y_i, \bar{x}, \bar{y}) \wedge \bigwedge_{j<m} \varphi_{n+j}^+(x_j, \bar{x}, \bar{y})$$

where $(n, n+m, \varphi_0(\bar{x}, \bar{y}), \ldots, \varphi_{n+m-1}(\bar{x}, \bar{y}))$ is a suitable tuple. Choose $\bar{a}$ in N with $N \models \varphi(\bar{a}, \bar{b})$. The elements of $\{a_0, \ldots, a_{m-1}, b_0, \ldots, b_{n-1}\}$ are pairwise different. W.l.o.g. $a_i \in N \setminus M$ for $i < m$. $N \models \varphi_i^+(b_i, \bar{a}, \bar{b})$ $(i < n)$ implies $N^{b_i} \models \varphi_i(\bar{a}, \bar{b})$. Since T is model-complete we have $M^{b_i} \preceq N^{b_i}$. Therefore $M^{b_i} \models \psi_{\varphi_i}(\bar{b})$ and $M \models (\psi_{\varphi_i})^+(b_i, \bar{b})$. By the axioms of T^1 we obtain $M \models \exists \bar{x}\, \varphi(\bar{x}, \bar{b})$, as desired. □

Corollary 2.7 *If T is model-complete and eliminates $\exists^\infty x_1$, then T^1 is model-complete and eliminates $\exists^\infty x_1$.*

Proof. We show that T^1 eliminates all quantifiers $\exists^\infty x_0 \ldots x_{m-1}$. Since T^1 is model-complete every formula is equivalent modulo T^1 to an existential formula. Note that

$$\models \exists^\infty \bar{x} \Big(\bigvee_{i<k} \varphi_i(\bar{x}, \bar{y}) \Big) \longleftrightarrow \bigvee_{i<k} (\exists^\infty \bar{x}\, \varphi_i(\bar{x}, \bar{y})).$$

Hence by Lemma 2.5 it is sufficient to consider the case $\exists^\infty \bar{x}\, \exists \bar{z}\, \varphi(\bar{x}, \bar{z}, \bar{y})$ where

$$\varphi(\bar{x}, \bar{z}, \bar{y}) \equiv \bigwedge_{i<n} \varphi_i^+(y_i, \bar{x}, \bar{z}, \bar{y}) \wedge \bigwedge_{i<\ell} \varphi_{n+i}^+(z_i, \bar{x}, \bar{z}, \bar{y}) \wedge \bigwedge_{i<m} \varphi_{n+\ell+i}^+(x_i, \bar{x}, \bar{z}, \bar{y})$$

and $(n, n+\ell+m, \varphi_0(\bar{x}, \bar{z}, \bar{y}), \ldots, \varphi_{n+\ell+m-1}(\bar{x}, \bar{z}, \bar{y}))$ is a suitable tuple. Then we use P. Winkler's idea behind Lemma 2.2:

$$\models \exists^\infty \bar{x}\, \exists z\, \psi(\bar{x}, z, \bar{y}) \longleftrightarrow \exists^\infty \bar{x} z\, \psi(\bar{x}, z, \bar{y}) \vee \exists z\, \exists^\infty \bar{x}\, \psi(\bar{x}, z, \bar{y}).$$

Hence we can assume w.l.o.g. that there is no existential quantifier $\exists \bar{z}$. Then $\varphi \equiv \bigwedge_{i<n} \varphi_i^+(y_i, \bar{x}, \bar{y}) \wedge \bigwedge_{i<m} \varphi_{n+i}^+(x_i, \bar{x}, \bar{y})$ where the $\varphi_i(\bar{x}, \bar{y})$ are conjunctions of L-literals and $(n, n+m, \varphi_0(\bar{x}, \bar{y}), \ldots, \varphi_{n+m-1}(\bar{x}, \bar{y}))$ is a suitable tuple. We claim that

$$\psi_\varphi(\bar{y}) \equiv \bigwedge_{i<n} (\psi_{\varphi_i})^+(y_i, \bar{y})$$

has the desired property:

$$M \models \psi_\varphi(\bar{b}) \quad \text{iff} \quad M \models \exists^\infty x_0 \ldots x_{m-1} \varphi(\bar{x}, \bar{b}).$$

This follows easily from the axioms of T^1. □

As above we define $T^{i+1} = (T^i)^1$. If we start with a model-complete theory T that eliminates the quantifier $\exists^\infty x_1$, then the theories $T^1, T^2, T^3, \ldots$ have these properties again.

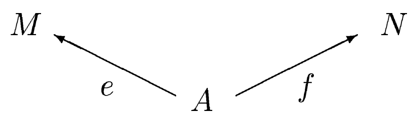
3 The algebraic closure

In this chapter we assume that T is countable and has no finite models. Often T is model-complete, and eliminates the quantifier $\exists^\infty x_1$. In many cases model-completeness will be replaced by stronger properties. First we have a surprise.

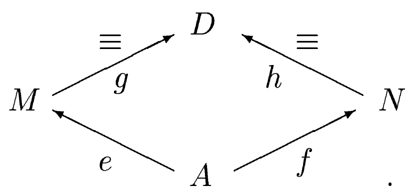
Theorem 3.1 *Assume T is model-complete and eliminates the quantifier $\exists^\infty x_1$. Then T^1 is complete.*

Proof. Assume M and N are models of T^1. Then it is easy to construct a T^+-model K that contains M and N as disjoint L^+-substructures. Since T^1 is the model companion of T^+ we can embed K into a model of T^1. Hence w.l.o.g. K is a model of T^1. By model-completeness of T^1 follows $M \equiv N$. $\square$

Definition T is complete over algebraically closed sets, if whenever



where M and N are T-models, e and f are embeddings, $e(A)$ is algebraically closed in M, and $f(A)$ is algebraically closed in N, then there are elementary embeddings g and h and an amalgam $D \models T$ over A such that



Lemma 3.2 *If T is complete over algebraically closed sets and we have the situation above, then we can choose g, h, and D in such a way that $g(b) = h(c)$ implies $b = e(a)$ for some $a \in A$.*

Proof. W.l.o.g. D is a big model in the sense of Hodges [H]. Then the lemma follows from the following well-known fact:

If $B \supseteq A \subseteq C$ are algebraically closed subsets of D, then there is an automorphism f of D such that $f(a) = a$ for $a \in A$ and

$$f(B) \cap C = A.$$

A nice proof of this statement due to Ph. Rothmaler and M. Ziegler can be found in the English version [R] of Rothmaler's model theory book. $\square$

Note that completeness over algebraically closed sets is implied by elimination of quantifiers.

Our aim is to show that completeness over algebraically closed sets is preserved by our construction.

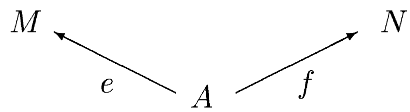
Definition Assume M is a model of T^+ and A a subset of M. Then $a \in \mathrm{acl}^*(A)$, if there is a finite sequence $a_0, a_1, \ldots, a_n = a$ such that, for $0 \leq i \leq n$

$$a_i \in A \text{ or there is some } j < i \text{ such that } a_i \in \mathrm{acl}_{M^{a_j}}(\{a_\ell : \ell < i\}).$$

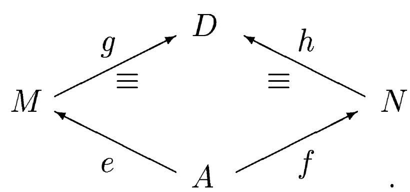
In the definition above $\mathrm{acl}_{M^{a_j}}(\ldots)$ is the algebraic closure in the T-model M^{a_j}. We have $\mathrm{acl}^*(A) \subseteq \mathrm{acl}(A)$. It makes sense to speak about completeness over acl^*-closed sets.

Theorem 3.3 *Assume T is model-complete, complete over algebraically closed sets, and eliminates $\exists^\infty x_1$. Then T^1 has these properties and* $\mathrm{acl}(A) = \mathrm{acl}^*(A)$.

Proof. By the assumptions T^1 is a model companion of T^+, is complete, and eliminates the quantifier $\exists^\infty x_1$. We consider



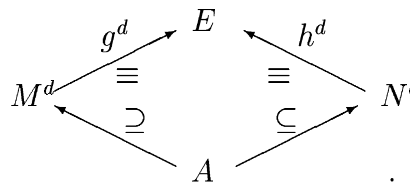
where M and N are models of T^1, e and f are L^+-embeddings, $e(A)$ is acl^*-closed in M and $f(A)$ is acl^*-closed in N. We show the existence of elementary embeddings g, h, and $D \models T^1$ such that



and $g(M) \cap h(N) = ge(A) = hf(A)$. From this follows immediately $\mathrm{acl}(A) = \mathrm{acl}^*(A)$, if we amalgamate M with itself over $\mathrm{acl}^*(A)$.
It is enough to show that there is $D \models T^+$ such that $M \subset_{L^+} D$ and $N \subset_{L^+} D$ in the case $\mathrm{dom}(M) \cap \mathrm{dom}(N) = \mathrm{dom}(A)$ and $e = f$ is the inclusion. By Lemma 2.3 we can embed D into a T^1-model. Hence $D \models T^1$ w.l.o.g. By model-completeness of T^1 g and h are elementary.
We have assumed that A is a common substructure of M and N: $\mathrm{dom}(M) \cap \mathrm{dom}(N) = \mathrm{dom}(A)$. Let $\mathrm{dom}(D)$ be a set that contains $\mathrm{dom}(M) \cup \mathrm{dom}(N)$, such that

$$|\mathrm{dom}(D) \setminus (\mathrm{dom}(M) \cup \mathrm{dom}(N))| = |\mathrm{dom}(D)|.$$

For every $d \in D$ we fix the structure of D^d. First assume $d \in A$. Since $\mathrm{acl}^T(A) = A$ in M^d and N^d by completeness of T over algebraically closed sets we have



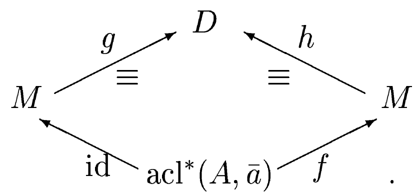
By Lemma 3.2 we can find E, g^d, h^d in such a way that $g^d(M^d) \cap h^d(N^d) = g^d(A)$. Note $\mathrm{dom}(M^d) = \mathrm{dom}(M)$ and $\mathrm{dom}(N^d) = \mathrm{dom}(N)$. W.l.o.g. we assume that $\mathrm{dom}(E) = \mathrm{dom}(D)$. To get D^d we put the structure of E onto $\mathrm{dom}(D)$ in such a way that $g^d(M^d)$ will be put onto $\mathrm{dom}(M)$ in the canonical way and $h^d(N^d)$ onto $\mathrm{dom}(N)$. For $d \in M \setminus A$ we define D^d as an elementary extension of M^d. For $d \in N \setminus A$ we define D^d as an elementary extension of N^d. Finally for $d \in D \setminus (M \cup N)$ D^d is any T model with domain $\mathrm{dom}(D)$. Then D has the desired properties. □

Corollary 3.4 *Assume that T is model-complete, complete over algebraically closed sets (e.g. T has quantifier elimination), and T eliminates $\exists^\infty x_1$. Then we have*

i) *Assume that $\bar{a}$, $\bar{b}$ and A are in M. Then* $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$ *if and only if there is an A-isomorphism of the L^+-structure* $\mathrm{acl}^*(A, \bar{a})$ *onto the L^+-structure* $\mathrm{acl}^*(A, \bar{b})$ *which carries $\bar{a}$ to $\bar{b}$.*

ii) *Now we replace completeness over algebraically closed sets by the stronger assumption that T has quantifier elimination. Modulo T^1 every formula $\varphi(\bar{x})$ is equivalent to a boolean combination of formulas $\exists \bar{z} \psi(\bar{x}, \bar{z})$ where $\psi(\bar{x}, \bar{z})$ is a conjunction of L^+-literals such that $\psi(\bar{a}, \bar{b})$ implies that $\bar{b} \subseteq \mathrm{acl}^*(\bar{a})$.*

Proof. i) By Theorem 3.3 T^1 is complete over algebraically closed sets. To show the non-trivial direction let f be the given A-isomorphism. Then we have



Hence $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$.

ii) We show that the formulas $\exists \bar{z} \psi(\bar{x}, \bar{z})$ described in ii) determine the isomorphism type of $\mathrm{acl}^*(\bar{a})$ in a T^1-model M. Then by i) they fix the type of $\bar{a}$. Hence every L^+-formula $\varphi(\bar{x})$ is equivalent to a boolean combination of such formulas modulo T^1. Assume $\bar{b} \subseteq \mathrm{acl}^*(\bar{a})$. After an enlargement of $\bar{b}$ we can assume w.l.o.g. that there are an enumeration $b_1, \ldots, b_n$ of $\bar{b}$ and formulas $\psi_a(\bar{x}, \bar{z})$ for $a \in \bar{a}$ and $\psi_{b_j}(\bar{x}, \bar{z})$ for $b_j \in \bar{b}$ in L such that

$$M \models \bigwedge_{a \in \bar{a}} \psi_a^+(a, \bar{a}, \bar{b}) \wedge \bigwedge_{1 \le j \le n} \psi_{b_j}^+(b_j, \bar{a}, \bar{b})$$

and for each j with $1 \leq j \leq n$ there is some $a \in \bar{a}$ such that $\exists z_{j+1} \ldots z_n \, \psi_a(\bar{a}, b_1, \ldots, b_{j-1}, z_j, \ldots, z_n)$ isolates algebraically $\mathrm{tp}(b_j/\bar{a}b_1, \ldots, b_{j-1})$ in M^a or there is some $i < j$ such that $\exists z_{j+1} \ldots z_n \, \psi_{b_i}(\bar{a}, b_1, \ldots, b_{j-1}, z_j, \ldots, z_n)$ isolates algebraically $\mathrm{tp}(b_j/\bar{a}b_1, \ldots, b_{j-1})$ in M^{b_i}. Since T has quantifier elimination we can choose the formulas above as conjunctions of literals. Hence a formula that isolates algebraically $\bar{b}$ over $\bar{a}$ in T^1 can be chosen of this form. Hence $\mathrm{acl}(\bar{a}) = \mathrm{acl}^*(\bar{a})$ is determined by these formulas. □

We consider the following property of T: $\mathrm{acl}(A) = A$ for all subsets A of models of T. We have $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$ for every n-ary function. Then T has quantifier elimination iff T is complete over algebraically closed sets.
Note that this property furthermore implies that T eliminates the quantifier $\exists^\infty x$. Lemma 2.1 gives the elimination of all quantifiers $\exists^\infty x_1 \ldots x_n$. We have the stronger result:

Lemma 3.5 *Assume* $\mathrm{acl}(A) = A$ *for all* $A \subseteq M \vDash T$. *Then we have for all models* M *of* T *and* $\bar{b}$ *in* M

$$M \vDash \exists x_1 \ldots x_n (\varphi(x_1, \ldots, x_n, \bar{b}) \wedge \bigwedge_{i,j} x_i \neq b_j)$$

implies

$$M \vDash \exists^\infty x_1 \ldots x_n \, \varphi(x_1, \ldots, x_n, \bar{b}).$$

Proof. To prove this we assume w.l.o.g. that $\varphi(\bar{x}, \bar{y})$ implies that two different variables stand for different elements. We use induction on n. For $n = 1$ the assertion is clear. Hence $M \vDash \exists x_1 \ldots x_n \, \varphi(x_1, \ldots, x_n, \bar{b})$ implies $M \vDash \exists^\infty x_1 \exists x_2 \ldots x_n \, \varphi(x_1, \ldots, x_n, \bar{b})$. Let $a_1^0, a_1^1, \ldots$ be different solutions of $\exists x_2 \ldots x_n \, \varphi(x_1, x_2, \ldots, x_n, \bar{b})$ in M. Then we apply the induction hypothesis to every $\varphi(a_1^i, x_2, \ldots, x_n, \bar{b})$. We use the obtained solutions, to ensure $M \vDash \exists^\infty x_1 \ldots x_n \, \varphi(x_1, \ldots, x_n, \bar{b})$. □

Corollary 3.6 *If* T *has elimination of quantifiers, and* $\mathrm{acl}(A) = A$ *for all* $A \subseteq M \vDash T$, *then the* T^n *have these properties.*

Proof. We have to show $\mathrm{acl}(A) = A$ for $A \subseteq M \vDash T^1$. But this follows from Theorem 3.3. □

Note that $\mathrm{acl}(A) \neq A$ for some $A \subseteq M \vDash T$ implies $\mathrm{acl}(B)$ is infinite for some finite $B \subseteq N \vDash T^1$. In this case T^1 is not $\aleph_0$-categorical. If T has finite signature, has elimination of quantifiers and $\mathrm{acl}(A) = A$ for $A \subseteq M \vDash T$, then T is $\aleph_0$-categorical. We have:

Corollary 3.7 *If* T *has elimination of quantifiers,* $\mathrm{acl}(A) = A$ *for all* $A \subseteq M \vDash T$, *and* T *is* $\aleph_0$*-categorical, then the* T^n *have these properties.* □

Examples for Corollary 3.7 are the theory of an unary infinite and coinfinite predicate and the theory of an equivalence relation with infinitely many classes and each class infinite.

4 Stability theory

Only in few cases T^1 is simple.

Theorem 4.1 *Assume that T has quantifier elimination,* $\mathrm{acl}(A) = A$ *for all $A \subseteq M \models T$, and for all A, B, C with $A \subseteq B$ in a model M of T* $\mathrm{tp}(C/B)$ *does not fork over A if and only if $C \cap B = C \cap A$. Then the T^n also have all these properties.*

Proof. By Corollary 3.6 it remains to consider forking in T^1. Let $\mathfrak{C}$ be a big model of T^1 and let A, B, C be sets in $\mathfrak{C}$ such that $A \subseteq B$ and $C \cap B = C \cap A$. It is sufficient to show that for all such A, B, C $\mathrm{tp}(C/B)$ does not divide over A. Then $\mathrm{tp}(C/B)$ does not fork over A, since for every $E \supseteq B$ there is some C^* such that $\mathrm{tp}(C^*/E)$ extends $\mathrm{tp}(C/B)$ and $C^* \cap E = C^* \cap A$. It follows $\mathrm{tp}(C^*/E)$ does not divide over A. We obtain C^* using compactness and Lemma 3.5.

W.l.o.g. C is finite. Let $B_0 = B, B_1, B_2, \ldots$ be an indiscernible sequence over A. Let f_i be an automorphism of $\mathfrak{C}$ that fixes A pointwise and sends B to B_i ($f_0 = \mathrm{id}$). We have to show that $\bigcup_{i<\omega} f_i(\mathrm{tp}(C/B))$ is consistent. Since the B_i are indiscernible over A there is a common subset D of the B_i such that $A \subseteq D$, D is fixed pointwise by each f_i, and the $D_i = B_i \setminus D$ are pairwise disjoint. Hence $D, D_0, D_1, \ldots$ are all pairwise disjoint. Let $x_1, \ldots, x_n$ be variables for the elements of C. Let G be any small subset of $\mathfrak{C}$. Since T^1 has quantifier elimination every type $p(x_1, \ldots, x_n) \in S_n(G)$ is completely determined by its L^+-literals $\psi^+(x_i, \bar{x}, \bar{g})$ and $\psi^+(g, \bar{x}, \bar{g})$ where $\psi^+(z, \bar{x}, \bar{y})$ comes from a L-literal $\psi(\bar{x}, \bar{y})$. We can restrict ourselves to conjunctions $\varphi(\bar{x}, \bar{y})$ of L-literals that satisfy the following condition:

$(*)$ $\varphi(\bar{x}, \bar{y})$ implies that the elements of $\bar{x}, \bar{y}$ are pairwise distinct.

Now we define

$$[p]^{x_i} = \{\varphi^+(x_i, \bar{x}, \bar{g}) : \varphi^+(x_i, \bar{x}, \bar{g}) \text{ is a conjunction of literals in } p \text{ and } \varphi \text{ satisfies } (*)\}$$

and for $g \in G$

$$[p]^g = \{\varphi^+(g, \bar{x}, \bar{g}) : \varphi^+(g, \bar{x}, \bar{g}) \text{ is a conjunction of literals in } p \text{ and } \varphi \text{ satisfies } (*)\}.$$

Let $Y, Y_0, Y_1, \ldots$ be sequences of new variables that are in bijective correspondence to $D, D_0, D_1, \ldots$ respectively. We use

$$[\mathrm{tp}(x_1 \ldots x_n/YY_i)]^{x_j} = \{\varphi^+(x_j, \bar{x}, \bar{y}, \bar{y}_i) : \varphi^+(x_j, \bar{x}, \bar{d}, \bar{d}_i) \in [f_i(\mathrm{tp}(C/B))]^{x_j}\}$$

and $\Big[\bigcup_{i<\omega} \mathrm{tp}(x_1 \ldots x_n/YY_i)\Big]^{x_j} = \bigcup_{i<\omega} [\mathrm{tp}(x_1 \ldots x_n/YY_i)]^{x_j}$.

By Lemma 3.5 and the axioms of T^1 it is sufficient to show that for $d \in D \cup \bigcup_{i<\omega} D_i$

$$\Big[\bigcup_{i<\omega} f_i(\mathrm{tp}(C/B))\Big]^d$$

is consistent and for $1 \leq j \leq n$

$$\Big[\bigcup_{i<\omega} \mathrm{tp}(x_1 \dots x_n / Y Y_i)\Big]^{x_j}$$

is consistent. For $d \in D_j$ we have

$$\Big[\bigcup_{i<\omega} f_i(\mathrm{tp}(C/B))\Big]^d = [f_j(\mathrm{tp}(C/B))]^d$$

and therefore it is consistent.
Now we consider $d \in D$. We have $C \cap B = C \cap A$ in $\mathfrak{C}^d$. By our assumption the complete type of T determined by $[\mathrm{tp}(C/B)]^d$ does not fork over A. Since $f_i(d) = d$ all f_i are automorphisms of $\mathfrak{C}^d$ that fix A pointwise. $B_0, B_1, B_2, \dots$ is an indiscernible sequence over A in $\mathfrak{C}^d$ since $d \in B_i$ for $i < \omega$. It follows

$$\bigcup_{i<\omega} f_i([\mathrm{tp}(C/B)]^d) = \bigcup_{i<\omega} [f_i(\mathrm{tp}(C/B))]^d = \Big[\bigcup_{i<\omega} f_i(\mathrm{tp}(C/B))\Big]^d$$

is consistent. Finally we look at

$$\Big[\bigcup_{i<\omega} \mathrm{tp}(x_1 \dots x_n / Y Y_i)\Big]^{x_j}.$$

Let $\psi^+(x_j, x_1, \dots, x_n, Y, Y_0)$ be a finite subset of $[\mathrm{tp}(x_1 \dots x_n / Y Y_0)]^{x_j}$. We can assume w.l.o.g. that $Y Y_0$ is finite. $\psi(\bar{x}, Y, Y_0)$ is a conjunction of literals with $(*)$ in L. By Lemma 3.5 $\exists Y_0 \psi(\bar{x}, Y, Y_0)$ implies $\exists^\infty Y_0 \psi(\bar{x}, Y, Y_0)$ in T. Hence $\bigcup_{i<\omega} \psi^+(x_j, x_1, \dots, x_n, Y, Y_i)$ is consistent. □

Theorem 4.1 provides us simple theories. For instance if T is the theory of some unary predicates, where each Boolean combination of them is either empty or infinite. On the other side if T is the theory of an equivalence relation with infinitely many classes we get that T^1 is not simple by the following argument:

Lemma 4.2 *Assume T has quantifier elimination and eliminates $\exists^\infty x$. If there are a formula $\varphi(x, \bar{y})$ and tuples $\bar{a}_0, \bar{a}_1, \dots$ in a model M of T such that the $\varphi(M, \bar{a}_i)$ are pairwise disjoint and infinite, then T^1 is not simple.*

Proof. W.l.o.g. we can assume that $\varphi(x, \bar{y})$ is a conjunction of literals. In T^1 $\varphi^+(z, x, \bar{y})$ has the tree property. □

Also stability is lost in most cases:

Lemma 4.3 *Assume that T has quantifier elimination and eliminates $\exists^\infty x$. If there is an infinite and coinfinite definable unary predicate $\psi(x, \bar{b})$ in a model of T, then T^1 has the independence property.*

Proof. W.l.o.g. we can assume that $\psi(x, \bar{b})$ is an atomic formula. $\psi^+(z, x, \bar{y})$ has the independence property. □

Theorem 4.4 *Let L be a relational language. Assume that T has quantifier elimination and eliminates $\exists^\infty x$. Furthermore assume that T^1 also has quantifier elimination. If T is stable, then T^1 does not have the strict order property.*

Proof. We assume that T^1 has the strict order property and show a contradiction. By [Sh] we can assume that there is a formula $\varphi(x, \bar{y})$ in L^+ that has the strict order property with respect to T^1. This means that there are tuples $\bar{a}_0, \bar{a}_1, \bar{a}_2, \dots$ in a big model $\mathfrak{C}$ of T^1 such that

$$\varphi(\mathfrak{C}, \bar{a}_s) \subsetneq \varphi(\mathfrak{C}, \bar{a}_r) \quad \text{for} \quad s < r.$$

By compactness we can assume

$$|\varphi(\mathfrak{C}, \bar{a}_r) \setminus \varphi(\mathfrak{C}, \bar{a}_s)| \geq \aleph_0.$$

We can find the parameters in such a way that

(1) $\{\bar{a}_0, \bar{a}_1, \bar{a}_2, \dots\}$ is an indiscernible sequence.

Let $\bar{a}_s$ be $(a_s^1, \dots, a_s^k)$. Using (1) we can assume w.l.o.g.

(2) There is some ℓ such that
$1 \leq \ell \leq k$,
$a_s^i \neq a_s^j$ for $i \neq j$,
$a_s^i = a_r^i$ for $i < \ell$ and $s, r < \omega$,
$a_s^i \neq a_r^j$ for $(i, s) \neq (j, r)$ and $\ell \leq j$.

Since we have a relational language and quantifier elimination for T^1 we can assume that $\varphi(x, \bar{y})$ has the form $\bigvee_{i<m} \varphi^i(x, \bar{y})$ where

$$(+) \qquad \varphi^i(x, \bar{y}) = \bigwedge_{1 \leq j \leq k} \psi_j^{i+}(y_j, x, \bar{y}) \wedge \psi_{k+1}^{i+}(x, x, \bar{y})$$

and the $\psi_j^i(x, \bar{y})$ are conjunctions of literals of L. By (2) we can define $a^j = a_s^j$ for all s and $1 \leq j < \ell$. W.l.o.g. we can assume that for $s < r$:

(3) There are infinitely many $c \in \mathfrak{C}$ such that $\mathfrak{C} \models \varphi^0(c, \bar{a}_r) \wedge \neg\big[\bigvee_{i<m} \varphi^i(c, \bar{a}_s)\big]$.

For every c that fulfils (3) there is a function f_c from $\{i : i < m\}$ into $\{j : 1 \leq j \leq k+1\}$ such that

$$\mathfrak{C} \models \neg\psi_{f(i)}^{i+}(a_s^{f(i)}, c, \bar{a}_s) \ \text{ for } \ f(i) < k+1$$

and

$$\mathfrak{C} \models \neg\psi_{k+1}^{i+}(c, c, \bar{a}_s) \ \text{ for } \ f(i) = k+1.$$

Conversely these conditions ensure that

$$\mathbb{C} \models \neg\Big[\bigvee_{i<m} \varphi^i(c, \bar{a}_s)\Big].$$

There are infinitely many $c \in \mathbb{C}$ that fulfil (3) and have a common function $f = f_c$. Hence (3) implies that

(4) there are infinitely many $c \in \mathbb{C}$ such that

a) for $1 \leq j < \ell$

$$\mathbb{C}^{a^j} \models \Big[\psi_j^0(c, \bar{a}_r) \wedge \bigwedge_{f(i)=j} \neg\psi_j^i(c, \bar{a}_s)\Big];$$

b) for $\ell \leq j \leq k$

$$C^{a_r^j} \models \psi_j^0(c, \bar{a}_r);$$

c) for $\ell \leq j \leq k$

$$C^{a_s^j} \models \neg\psi_j^i(c, \bar{a}_s) \text{ for all } i \text{ with } f(i) = j$$

d) $\mathbb{C}^c \models \psi_{k+1}^0(c, \bar{a}_r) \wedge \bigwedge_{f(i)=k+1} \neg\psi_{k+1}^i(c, \bar{a}_s)$.

We use that T and T^1 eliminate $\exists^\infty x$. (4) implies:

(5) a) For $1 \leq j < \ell$

$$\mathbb{C} \models \exists^\infty x(\psi_j^{0+}(a^j, x, \bar{a}_r) \wedge \bigwedge_{f(i)=j} \neg\psi_j^{i+}(a^j, x, \bar{a}_s)).$$

b) For $\ell \leq j \leq k$

$$\mathbb{C} \models \exists^\infty x\, \psi_j^{0+}(a_r^j, x, \bar{a}_r).$$

c) For $\ell \leq j \leq k$

$$\mathbb{C} \models \exists^\infty x \bigwedge_{f(i)=j} \neg\psi_j^{i+}(a_s^j, x, \bar{a}_s).$$

d) $\exists x \exists \bar{y}_r \bar{y}_s \Big[\psi_{k+1}^0(x, \bar{y}_r) \wedge \bigwedge_{f(i)=k+1} \neg\psi_{k+1}^i(x, \bar{y}_s)\Big]$ is consistent with T.

By the axioms (5) implies

$$\mathbb{C} \models \exists^\infty x\, \varphi(x, \bar{a}_r) \wedge \neg\varphi(x, \bar{a}_s).$$

If we fix j in (5)a), then we have a statement in $\mathbb{C}^{a^j}$. In $\mathbb{C}^{a^j}$ our sequence $\bar{a}_0, \bar{a}_1, \ldots$ is an indiscernible sequence, since $a^j = a^j_s$ for all s.
By assumption T is stable. This implies that $\bar{a}_0, \bar{a}_1, \ldots$ is an indiscernible set in $\mathbb{C}^{a^j}$. Hence we can exchange the roles of $\bar{a}_s$ and $\bar{a}_r$ and obtain:

(6a) For $1 \leq j < \ell$

$$\mathbb{C} \models \exists^\infty x(\psi_j^{0+}(a^j, x, \bar{a}_s) \wedge \bigwedge_{f(i)=j} \neg\psi_j^i(a^j, x, \bar{a}_r)).$$

Since $\bar{a}_0, \bar{a}_1, \ldots$ is an indiscernible sequence in $\mathbb{C}$ $(T^1!)$ we have:

(6b) For $\ell \leq j \leq k$

$$\mathbb{C} \models \exists^\infty x \psi_j^{0+}(a^j_s, x, \bar{a}_s).$$

(6c) For $\ell \leq j \leq k$

$$\mathbb{C} \models \exists^\infty x \bigwedge_{f(i)=j} \neg\psi_j^{i+}(a^j_r, x, \bar{a}_r).$$

If we change names we can write (5)d) in the form

(6d) $\exists x \exists \bar{y}_s \bar{y}_r \Big[\psi^0_{k+1}(x, \bar{y}_s) \wedge \bigwedge_{f(i)=k+1} \neg\psi^i_{k+1}(x, \bar{y}_r)\Big]$ is consistent with T.

Now (6) implies

$$\mathbb{C} \models \exists^\infty x\, \varphi(x, \bar{a}_s) \wedge \neg\varphi(x, \bar{a}_r),$$

the desired contradiction. □

Corollary 4.5 *If T has quantifier elimination, eliminates $\exists^\infty x$, $\mathrm{acl}(A) = A$ for all $A \subseteq M \models T$, and is stable, then T^1 does not have the strict order property.*

Proof. The assumptions imply quantifier elimination for T^1 and (+). The restriction to a relational language in Theorem 4.4 is only used to get (+). □

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